

INVOLUTIONS AND HIGHER ORDER AUTOMORPHISMS OF HIGGS BUNDLE MODULI SPACES

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ABSTRACT. We consider the moduli space $\mathcal{M}(G)$ of G -Higgs bundles over a compact Riemann surface X , where G is a complex semisimple Lie group. This is a hyperkähler manifold homeomorphic to the moduli space $\mathcal{R}(G)$ of representations of the fundamental group of X in G . In this paper we first study holomorphic involutions of $\mathcal{M}(G)$ defined by combining elements of order 2 in $H^1(X, Z) \rtimes \text{Out}(G)$, where Z is the centre of G and $\text{Out}(G)$ is the group of outer automorphisms of G , with multiplication of the Higgs field by ± 1 , and describe the subvarieties of fixed points. These are hyperkähler submanifolds of $\mathcal{M}(G)$ in the (+1)-case, corresponding to the moduli of representations of the fundamental group in certain reductive complex subgroups of G defined by holomorphic involutions of G ; while in the (-1)-case they are Lagrangian subvarieties corresponding to the moduli of representations of the fundamental group of X in real forms of G and certain extensions of these. We go on to study higher order automorphisms of $\mathcal{M}(G)$ obtained by combining the action of an element of order n in $H^1(X, Z) \rtimes \text{Out}(G)$ with the multiplication of the Higgs field by an n th-root of unity.

1. INTRODUCTION

Let G be a complex connected semisimple Lie group with Lie algebra \mathfrak{g} . Let X be a smooth projective curve over \mathbb{C} , equivalently a compact Riemann surface. A G -Higgs bundle over X is a pair (E, φ) where E is a principal G -bundle over X and φ is a section of $E(\mathfrak{g}) \otimes K$, where $E(\mathfrak{g})$ is the bundle associated to E via the adjoint representation of G , and K is the canonical bundle on X . These objects were introduced by Hitchin [24, 25]. There are notions of (semi)stability and polystability for G -Higgs bundles which allow to consider the moduli space of polystable G -Higgs bundles $\mathcal{M}(G)$, which has the structure of a complex algebraic variety. Let Z be the centre of G and let $\text{Out}(G)$ be the group of outer automorphisms of G . The group $H^1(X, Z) \rtimes \text{Out}(G) \times \mathbb{C}^*$ acts naturally on $\mathcal{M}(G)$. In this paper we study involutions and higher order automorphisms defined by elements of this group and describe their fixed points. The simplest involution $(E, \varphi) \mapsto (E, -\varphi)$ was already studied by Hitchin when $G = \text{SL}(2, \mathbb{C})$ in [24].

An important feature of G -Higgs bundles is their relation with representations of the fundamental group of X in G . By a representation we mean a homomorphism $\rho : \pi_1(X) \rightarrow G$. We say that ρ is reductive if the Zariski closure of the image of ρ in G is a reductive group, or equivalently, if the composition of ρ with the adjoint representation of G gives a totally reducible representation of $\pi_1(X)$ in \mathfrak{g} . The set of reductive representations in

2000 *Mathematics Subject Classification.* Primary 14H60; Secondary 57R57, 58D29.

Partially supported by the Spanish MINECO under the ICMAT Severo Ochoa grant No. SEV-2011-0087, and grant No. MTM2013-43963-P and by the European Commission Marie Curie IRSES MODULI Programme PIRSES-GA-2013-612534.

G modulo conjugation by G , which we will denote by $\mathcal{R}(G)$, is called the moduli space of representations of $\pi_1(X)$ in G , and has the structure of a complex algebraic variety. Non-abelian Hodge theory establishes that $\mathcal{M}(G)$ and $\mathcal{R}(G)$ are homeomorphic. This is proved by combining results by Hitchin [24] and Donaldson [13] for $\mathrm{SL}(2, \mathbb{C})$ and Simpson [44, 45] and Corlette [11] in general. In this paper we establish the relation of the fixed points of the involutions on $\mathcal{M}(G)$ mentioned above with representations of $\pi_1(X)$ in G .

An important ingredient in the theory of Higgs bundles is the \mathbb{C}^* action on $\mathcal{M}(G)$ defined by $(E, \varphi) \mapsto (E, \lambda\varphi)$ for $\lambda \in \mathbb{C}^*$. In particular, multiplication by -1 sending (E, φ) to $(E, -\varphi)$ defines the simplest non-trivial involution in $\mathcal{M}(G)$ that we consider in this paper. To describe the fixed points, consider the group $\mathrm{Int}(G)$ of inner automorphisms of G , and an element $\theta \in \mathrm{Int}(G)$ of order 2. Then θ gives an involution of \mathfrak{g} which can be decomposed in terms of ± 1 -eigenspaces as $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$, where clearly \mathfrak{g}^+ is the Lie algebra of G^θ , the subgroup of fixed points under θ . The restriction of the adjoint representation of G to G^θ gives the adjoint representation of G^θ in \mathfrak{g}^+ as well as the isotropy representation $G^\theta \rightarrow \mathrm{GL}(\mathfrak{g}^-)$. The fixed points of our involution $(E, \varphi) \mapsto (E, -\varphi)$ are given by pairs (E, φ) consisting of a G^θ -bundle E and $\varphi \in H^0(X, E(\mathfrak{g}^-) \otimes K)$, where $E(\mathfrak{g}^-)$ is the bundle associated to the isotropy representation. Two involutions $\theta, \theta' \in \mathrm{Int}_2(G)$ which are equivalent, in the sense that there is an $\alpha \in \mathrm{Int}(G)$ such that $\theta' = \alpha\theta\alpha^{-1}$, define the same locus in the fixed point set in $\mathcal{M}(G)$. So the fixed point set is determined basically by the set $\mathrm{Int}_2(G)/\sim$, where $\mathrm{Int}_2(G)$ is the set of elements of order 2 in $\mathrm{Int}(G)$ and \sim is the equivalence relation defined above.

In [9] Cartan shows that there exists a conjugation τ of G , i.e. an antiholomorphic involution, defining a maximal compact subgroup of G , such that in each class in $\mathrm{Int}_2(G)/\sim$ there is a θ commuting with τ . For such θ , we can consider the conjugation of G defined by $\sigma := \theta\tau$. The fixed point subgroup G^σ defines a real form of G , which is inner equivalent to the compact real form, what is known as a real form of Hodge type. The fixed points of the involution $(E, \varphi) \mapsto (E, -\varphi)$ correspond precisely with the representations of $\pi_1(X)$ in G^σ .

Of course, in general, there are other real forms which are not of Hodge type. In this paper we show that representations of $\pi_1(X)$ in these other real forms are also related to fixed points of appropriate involutions of $\mathcal{M}(G)$. To define these involutions, we consider the group $\mathrm{Aut}(G)$ of holomorphic automorphisms of G . The natural action of $\mathrm{Aut}(G)$ on $\mathcal{M}(G)$ descends to an action of the outer group of automorphisms $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$. In particular we can consider the set $\mathrm{Out}_2(G)$ of elements of order 2 in $\mathrm{Out}(G)$ and define for $a \in \mathrm{Out}_2(G)$ the involutions $\iota(a, \pm) : \mathcal{M}(G) \rightarrow \mathcal{M}(G)$ given by $(E, \varphi) \mapsto (a(E), \pm a(\varphi))$. We can extend the equivalence relation defined above to the set $\mathrm{Aut}_2(G)$ of elements of order 2 of $\mathrm{Aut}(G)$, establishing for $\theta, \theta' \in \mathrm{Aut}_2(G)$ that $\theta \sim \theta'$ if there is an $\alpha \in \mathrm{Int}(G)$ such that $\theta' = \alpha\theta\alpha^{-1}$. This equivalence descends to define a surjective map $\mathcal{C} : \mathrm{Aut}_2(G)/\sim \rightarrow \mathrm{Out}_2(G)$. Now, for $\theta \in \mathrm{Aut}_2(G)$, as in the case of $\theta \in \mathrm{Int}_2(G)$, we have the decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ and the representations $G^\theta \rightarrow \mathrm{GL}(\mathfrak{g}^\pm)$. The fixed points of $\iota(a, \pm)$ are given by (E, φ) where E is a G^θ -bundle, $\varphi \in H^0(X, E(\mathfrak{g}^\pm) \otimes K)$, and $[\theta] \in \mathcal{C}^{-1}(a)$. As in the previous case, involutions defining the same class $[\theta]$ determine the same locus in $\mathcal{M}(G)$. To relate to real forms of G , we first note that Cartan's result mentioned above applies in fact more generally to $\mathrm{Aut}_2(G)/\sim$

and hence in each class we can find a representative θ commuting with the compact conjugation τ , such that $\sigma := \theta\tau$ is a conjugation of G defining a real form G^σ . We will say that σ , and the corresponding θ are in the ‘clique’ a . The real forms of Hodge type are thus the ones corresponding to the trivial clique in $\text{Out}_2(G)$. Now, the fixed points of $\iota(a, +)$ are given by G^θ -Higgs bundles, thus corresponding to representations of $\pi_1(X)$ in G^θ , where θ is in the clique a , and hence defining hyperkähler subvarieties of $\mathcal{M}(G)$. The fixed points of $\iota(a, -)$ correspond to representations of $\pi_1(X)$ in G^σ , where σ is in the clique a , and define complex Lagrangian subvarieties of $\mathcal{M}(G)$, with respect to the complex structure of $\mathcal{M}(G)$ determined by the complex structure of X and the corresponding holomorphic symplectic structure.

We go a step further and consider the action of $H^1(X, Z)$, where Z is the centre of G , on $\mathcal{M}(G)$, defined by ‘tensoring’ a principal G -bundle by an element in $H^1(X, Z)$ — the principal bundle analogue of tensoring a vector bundle by a line bundle. We then consider elements $(\alpha, a) \in H^1(X, Z) \rtimes \text{Out}(G)$ of order 2, where the semidirect product is with respect to the natural action of $\text{Out}(G)$ on $H^1(X, Z)$, and define the involutions $\iota(a, \alpha, \pm)$ of $\mathcal{M}(G)$ given by $(E, \varphi) \mapsto (a(E) \otimes \alpha, \pm a(\varphi))$. The fixed points of these involutions are now related to what we call (G_θ, \pm) -Higgs bundles. These are pairs (E, φ) consisting of a G_θ -bundle E , where $\theta \in \text{Aut}_2(G)$ is in the clique of a and

$$G_\theta := \{g \in G : \theta(g) = c(g)g, \text{ with } c(g) \in Z\},$$

and $\varphi \in H^0(X, E(\mathfrak{g}^\pm) \otimes K)$.

The $(G_\theta, +)$ -bundles correspond with representations of $\pi_1(X)$ in G_θ , while the $(G_\theta, -)$ -bundles correspond with representations of $\pi_1(X)$ in the subgroup

$$G_\sigma := \{g \in G : \sigma(g) = c(g)g, \text{ with } c(g) \in Z\}.$$

The groups G_θ and G_σ are the normalizers of G^θ and G^σ , respectively, in G , and are extensions of G^θ and G^σ , respectively, by a finite group — the same in both cases. Again the fixed points subvarieties for $\iota(a, \alpha, +)$ are hyperkähler submanifolds, while those for $\iota(a, \alpha, -)$ are Lagrangian submanifolds.

We go on to generalise our study to automorphisms of $\mathcal{M}(G)$ defined by elements of $H^1(X, Z) \rtimes \text{Out}(G) \times \mathbb{C}^*$ of arbitrary finite order. For this, we take an element $(\alpha, a) \in H^1(X, Z) \rtimes \text{Out}(G)$ of order n and $\zeta_k := \exp(2\pi i \frac{k}{n})$, with $0 \leq k \leq n-1$, and consider the automorphism $\iota(a, \alpha, \zeta_k)$ of $\mathcal{M}(G)$ given by $(E, \varphi) \mapsto (a(E) \otimes \alpha, \zeta_k a(\varphi))$. The fixed points of this automorphism are described by what we call (G_θ, ζ_k) -Higgs bundles. These are pairs (E, φ) consisting of a G_θ -bundle E , where $\theta \in \text{Aut}_n(G)$ is in the class defined by a and, as above,

$$G_\theta := \{g \in G : \theta(g) = c(g)g, \text{ with } c(g) \in Z\},$$

and $\varphi \in H^0(X, E(\mathfrak{g}^k) \otimes K)$, where \mathfrak{g}^k is the eigenspace with eigenvalue ζ_k of the automorphism of \mathfrak{g} defined by θ . Except for the case $k = 0$, which corresponds to ordinary G_θ -Higgs bundles since \mathfrak{g}^0 is the Lie algebra of G_θ , the other cases are not generally related to representations of the fundamental group. The fixed points for $k = 0$ define indeed hyperkähler subvarieties of $\mathcal{M}(G)$.

The paper is organized as follows. In Section 2 we give the necessary background on involutions and real forms of complex semisimple Lie algebras and Lie groups. Somethings are quite standard, but we include other relevant results necessary for our Higgs bundle

analysis that we have not found in the literature. Recently we learned of the paper by J. Adams [1] which relates closely to the Galois cohomology approach to real forms that we take in this paper. We extend some of the features to arbitrary finite order automorphisms of a complex semisimple Lie group G . Since the arguments are basically the same as those for involutions, we mostly avoid details on the proofs. In Section 3 we study finite order automorphisms of a G -bundle and show how these give rise to reductions of structure group of the bundle. We consider a version of this, twisted by a finite order automorphism of G and a subgroup of $Z(G)$. These will play a central role in the study of our involutions and higher order automorphisms of the moduli space of G -Higgs bundles.

In Section 4 we review the basics of G -Higgs bundle theory, where G is a complex semisimple Lie group, including the notions of stability and the Hitchin–Kobayashi correspondence, relating to solutions to the Hitchin equations. This allows us to show the moduli space $\mathcal{M}(G)$ appearing as a hyperkähler quotient. We then describe a natural group of automorphisms of $\mathcal{M}(G)$, which in particular will provide with the involutions and higher order automorphisms that we study in this paper. In Section 5 we introduce a class of G -Higgs bundles with an extra structure determined by an element $\theta \in \text{Aut}_2(G)$. These are the (G^θ, \pm) - and (G_θ, \pm) -Higgs bundles mentioned above. These objects require appropriate stability conditions, defining moduli spaces $\mathcal{M}(G^\theta, \pm)$ and $\mathcal{M}(G_\theta, \pm)$, in terms of which we will describe the fixed points of the involutions studied later. We extend this to elements $\theta \in \text{Aut}_n(G)$, defining the (G_θ, ζ_k) -Higgs bundles mentioned above.

In Section 6 we review first the basics on the moduli space $\mathcal{R}(G)$ of reductive representations of $\pi_1(X)$ in G , and its homeomorphism with the moduli space of G -Higgs bundles $\mathcal{M}(G)$. We also show how the moduli space of representations $\mathcal{R}(G)$, or equivalently the moduli space of flat G -connections is also a hyperkähler quotient in a natural way. We then show similar correspondences for $\mathcal{M}(G^\theta, \pm)$ when θ is of order 2 with the moduli space of representations $\mathcal{R}(G^\theta)$ in G^θ in the $(G^\theta, +)$ -case, and the moduli of representations $\mathcal{R}(G^\sigma)$ in G^σ in $(G^\theta, -)$ -case. Similarly for the moduli $\mathcal{M}(G_\theta, \pm)$. When θ is of order n , the relation to representations of $\pi_1(X)$ applies generally only for the moduli space of (G_θ, ζ_k) -Higgs bundles with $k = 0$.

In Section 7 we undertake the study of the involutions considered above on the moduli space of G -Higgs bundles. Building upon results in Sections 2, 3, 4 and 5, we describe the fixed point subvarieties. Our main results are Theorems 7.3, 7.6 and 7.11. To the cost of having a certain amount of repetition, we have chosen here to proceed step by step in terms of generality. Although Theorem 7.3 is a particular case of Theorem 7.6, which in turn follows from Theorem 7.11, we believe that our approach, dealing gradually with the various difficulties, makes our presentation clearer. In Section 8 we study how the involutions on $\mathcal{M}(G)$ translate to the moduli space $\mathcal{R}(G)$ of representations of the fundamental group of X in G . Here we use the non-abelian Hodge theory correspondences given in Section 6. Our main results are Theorem 8.3, 8.5, and 8.7. In the latter we establish the hyperkähler or Lagrangian property of the fixed point subvarieties according to the type of involution.

In Section 9 we illustrate our main results in Sections 7 and 8 to the case $G = \text{SL}(n, \mathbb{C})$. It is worth pointing out that, for $G = \text{SL}(2, \mathbb{C})$, the description of the fixed points involving elements of order 2 in $H^1(X, Z)$, which in this case can be identified with $J_2(X)$ — the 2-torsion elements in the Jacobian of X —, involves the Prym variety of the étale cover of

X defined by the element in $J_2(X)$. A detailed study of this case is carried out in [19]. For $\mathrm{SL}(n, \mathbb{C})$, with even n , there is a similar phenomenon which involves now generalised Prym varieties in the sense of Narasimhan–Ramanan [33]. This case and the general construction for arbitrary groups is being pursued in [20].

In Section 10 we extend our theory to automorphisms of $\mathcal{M}(G)$ defined by order n elements of $H^1(X, Z) \rtimes \mathrm{Out}(G) \times \mathbb{C}^*$. Our main results are Theorems 10.1 and 10.3, for which the proofs are straightforward generalisations of those of Theorems 7.6 and 7.11, respectively, and hence we omit. We finish in Section 11 applying our results to the study of involutions and order 3 automorphisms of the moduli space of $\mathrm{Spin}(8, \mathbb{C})$ -Higgs bundles and exploring the role of triality.

The main results of this paper have been presented in several conferences and workshops since 2006, and announced in [14, 15, 38]. We apologize for having taken so long to produce the final paper.

Acknowledgements. We wish to thank the following institutions for hospitality and support: MSRI, Berkeley; Newton Institute for Mathematical Sciences, Cambridge; Mathematical Institute, Oxford; NUS, Singapore; IISc, Bangalore; CMI, Chennai; ICMAT, Madrid.

2. AUTOMORPHISMS OF COMPLEX LIE GROUPS, INVOLUTIONS AND REAL FORMS

We recall some basic facts about automorphisms and real forms of complex Lie algebras and complex Lie groups (see [23, 36, 37]), and give some results needed for our analysis that we have not found in the literature.

2.1. Automorphisms of Lie algebras and Lie groups. Let \mathfrak{g} be a complex Lie algebra. We consider the linear algebraic group $\mathrm{Aut}(\mathfrak{g})$ of all automorphisms of \mathfrak{g} . The group $\mathrm{Int}(\mathfrak{g})$ of **inner automorphisms** of \mathfrak{g} is the normal subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by all elements of the form $\exp(\mathrm{ad} x)$, where ad is the adjoint representation of \mathfrak{g} and $x \in \mathfrak{g}$. The quotient

$$\mathrm{Out}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g}) / \mathrm{Int}(\mathfrak{g})$$

is called the group of **outer automorphisms** of \mathfrak{g} . We thus have an extension

$$(2.1) \quad 1 \longrightarrow \mathrm{Int}(\mathfrak{g}) \longrightarrow \mathrm{Aut}(\mathfrak{g}) \longrightarrow \mathrm{Out}(\mathfrak{g}) \longrightarrow 1.$$

Let G be a complex Lie group with Lie algebra \mathfrak{g} . If G is connected then $\mathrm{Int}(\mathfrak{g}) = \mathrm{Ad}(G)$, where Ad is the adjoint representation of G in \mathfrak{g} . Recall that $\mathrm{Ad}(G) \cong G/Z(G)$, where $Z(G)$ is the centre of G .

If \mathfrak{g} is semisimple, $\mathrm{Out}(\mathfrak{g})$ is isomorphic to the group of automorphisms of its Dynkin diagram. From this one may list the groups $\mathrm{Out}(\mathfrak{g})$ for the simple complex Lie algebras in Table 1.

Let G be a complex Lie group. Let $\mathrm{Aut}(G)$ be the group of holomorphic automorphisms of G and $\mathrm{Int}(G)$ be the normal subgroup of $\mathrm{Aut}(G)$ given by **inner automorphisms**. The elements of $\mathrm{Int}(G)$ will be denoted by $\mathrm{Int}(g)$ for $g \in G$ and the action on G is given by

$$\mathrm{Int}(g)(h) := ghg^{-1} \quad \text{for every } h \in G.$$

Then if G is connected $\text{Int}(G) \cong \text{Ad}(G)$. Let $\text{Out}(G) := \text{Aut}(G)/\text{Int}(G)$ be the group of **outer automorphisms** of G . It is well-known that if G is a connected complex reductive Lie group the extension

$$(2.2) \quad 1 \longrightarrow \text{Int}(G) \longrightarrow \text{Aut}(G) \longrightarrow \text{Out}(G) \longrightarrow 1.$$

splits (see [43]).

Let \tilde{G} be the universal cover of G . We clearly have

$$\text{Int}(G) \cong \text{Int}(\tilde{G}) \cong \text{Int}(\text{Ad}(G)) \cong \text{Ad}(G) \cong \text{Int}(\mathfrak{g}).$$

We thus observe that, if $\pi_1(G) = 1$ or $Z(G) = 1$, extension (2.2) is isomorphic to (2.1).

\mathfrak{g}	$\text{Out}(\mathfrak{g})$	$Z(\tilde{G})$
$A_n, n > 1$	$\mathbb{Z}/2$	$\mathbb{Z}/(n+1)$
A_1	$\{1\}$	$\mathbb{Z}/2$
B_n	$\{1\}$	$\mathbb{Z}/2$
C_n	$\{1\}$	$\mathbb{Z}/2$
D_4	S_3	$\mathbb{Z}/2 \times \mathbb{Z}/2$
$D_n, n > 4, n \text{ even}$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$
$D_n, n > 4, n \text{ odd}$	$\mathbb{Z}/2$	$\mathbb{Z}/4$
E_6	$\mathbb{Z}/2$	$\{1\}$
E_7	$\{1\}$	$\{1\}$
E_8	$\{1\}$	$\{1\}$
F_4	$\{1\}$	$\{1\}$
G_2	$\{1\}$	$\{1\}$

TABLE 1. Group of outer automorphisms and centres

Observe that $\text{Aut}(G)$ acts on $Z(G)$, defining an action of $\text{Out}(G)$ since $\text{Int}(G)$ acts trivially on $Z(G)$. Hence in order to compute $\text{Out}(G)$ for any semisimple complex Lie group G out of the Table 1, we need to compute how $\text{Out}(\tilde{G})$ acts on $Z(\tilde{G})$. These are listed when G is simple in Table 1 (see [22]). If a semisimple group G is isomorphic to \tilde{G}/Z' , where $Z' \subset Z(\tilde{G})$ is a subgroup, where \tilde{G} is the universal covering of G , we have $\pi_1(G) = Z(\tilde{G})/Z'$. Let $f \in \text{Aut}(\tilde{G})$. Then, clearly f descends to give an element in $\text{Aut}(G)$ if and only if $f(Z') \subset Z'$. Similarly, $\text{Out}(\tilde{G})$ acts on $Z(\tilde{G})$ and $a \in \text{Out}(\tilde{G})$ descends to an element in $\text{Out}(G)$ if Z' is invariant under a .

2.2. Involutions and real forms of a complex Lie algebra. Let \mathfrak{g} be a complex Lie algebra and $\mathfrak{g}_{\mathbb{R}}$ its underlying real Lie algebra. A real subalgebra \mathfrak{g}_0 of $\mathfrak{g}_{\mathbb{R}}$ is called a **real form** of \mathfrak{g} if $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. In this case $\mathfrak{g}_0 \otimes \mathbb{C}$ may be naturally identified with \mathfrak{g} . Given a real form \mathfrak{g}_0 , there is a bijection of $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ onto itself defined by

$$\sigma(x + iy) = x - iy, \quad x, y \in \mathfrak{g}_0,$$

which is a **conjugation or antiinvolution** of \mathfrak{g} , i.e. an antilinear homomorphism σ such that $\sigma^2 = \text{Id}$. Conversely, any conjugation $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$ defines the real form

$$\mathfrak{g}^\sigma := \{z \in \mathfrak{g} : \sigma(z) = z\}.$$

There is thus a bijective correspondence between real forms and conjugations of \mathfrak{g} .

To classify real forms of a complex Lie algebra \mathfrak{g} up to isomorphism, one first observes that if \mathfrak{g}_0 and \mathfrak{g}'_0 are two real forms of \mathfrak{g} and $f : \mathfrak{g}_0 \rightarrow \mathfrak{g}'_0$ is an isomorphism, then f extends uniquely to an automorphism $\alpha := f^\mathbb{C}$ of \mathfrak{g} . Now, if σ and σ' are the corresponding conjugations for \mathfrak{g}_0 and \mathfrak{g}'_0 respectively, clearly the conjugations $\alpha\sigma_0$ and $\sigma'_0\alpha$ coincide in \mathfrak{g}_0 . Therefore they coincide in \mathfrak{g} and hence $\sigma' = \alpha\sigma\alpha^{-1}$. Conversely, suppose that there exists $\alpha \in \text{Aut}(\mathfrak{g})$ such that $\sigma' = \alpha\sigma\alpha^{-1}$. It is immediate that $\alpha(\mathfrak{g}^\sigma) = \mathfrak{g}^{\sigma'}$, i.e. $\alpha(\mathfrak{g}_0) = \mathfrak{g}'_0$.

In other words, if $\text{Conj}(\mathfrak{g})$ is the set of conjugations of \mathfrak{g} , the set of isomorphism classes of real forms of \mathfrak{g} is in bijection with

$$(2.3) \quad \text{Conj}(\mathfrak{g}) / \sim_c,$$

where the equivalence relation \sim_c for $\sigma, \sigma' \in \text{Conj}(\mathfrak{g})$ is defined by

$$\sigma \sim_c \sigma' \text{ if there is } \alpha \in \text{Aut}(\mathfrak{g}) \text{ such that } \sigma' = \alpha\sigma\alpha^{-1}.$$

In [9] É. Cartan proved that for a semisimple complex Lie algebra \mathfrak{g} in the statement above one can replace conjugations (antiinvolutions) of \mathfrak{g} by \mathbb{C} -linear **involutions**, i.e. involutive automorphisms of \mathfrak{g} . This is based on the existence of a compact real form of \mathfrak{g} .

Let \mathfrak{g} be a complex Lie algebra and τ a fixed conjugation. We have a map

$$(2.4) \quad \begin{aligned} \text{Conj}(\mathfrak{g}) &\rightarrow \text{Aut}(\mathfrak{g}) \\ \sigma &\mapsto \theta := \sigma\tau. \end{aligned}$$

This correspondence depends on the choice of τ . Since $\sigma^2 = \text{Id}$, $\theta^{-1} = \tau\theta\tau$ and $\theta^2 = \text{Id}$ if and only if $\sigma\tau = \tau\sigma$. Also if θ corresponds to σ and $\alpha \in \text{Aut}(\mathfrak{g})$, then the automorphism θ' corresponding to $\alpha\sigma\alpha^{-1}$ has the form $\theta' = \alpha\theta(\tau\alpha\tau)^{-1}$.

We can rephrase these properties in terms of the Galois cohomology of $\mathbb{Z}/2$ in the group $\text{Aut}(\mathfrak{g})$. To explain this, we recall first the basic definition of **non-abelian cohomology** (see [42, Ch. III]). Let Γ be a group and A another group acted on by Γ , that is, every $\gamma \in \Gamma$ defines an automorphism of A that we will denote also by γ , such that $\gamma(xy) = \gamma(x)\gamma(y)$, for every $x, y \in A$.

We will define H^0 and H^1 . We set $H^0(\Gamma, A) := A^\Gamma$, the subgroup of elements of A fixed under Γ . To define H^1 we first define a 1-**cocycle** (or simply **cocycle**) of Γ in A as a map $\gamma \mapsto a_\gamma$ of Γ to A such that

$$(2.5) \quad a_{\gamma\gamma'} = a_\gamma\gamma(a_{\gamma'}) \text{ for } \gamma, \gamma' \in \Gamma.$$

The set of cocycles is denoted by $Z^1(\Gamma, A)$. Two cocycles $a, a' \in Z^1(\Gamma, A)$ are said to be **cohomologous** if there is $b \in A$ such that

$$(2.6) \quad a'_\gamma = b^{-1}a_\gamma\gamma(b).$$

This is an equivalence relation in $Z^1(\Gamma, A)$ and the quotient is denoted by $H^1(\Gamma, A)$. This is the **first cohomology set of Γ in A** . It has a distinguished element (called the "neutral

element") even though there is in general no composition: the class of the unit cocycle. If A is commutative $H^0(\Gamma, A)$ and $H^1(\Gamma, A)$ are the usual cohomology groups of dimensions 0 and 1.

We will be dealing with the special case in which $\Gamma = \{1, \gamma\} \cong \mathbb{Z}/2$, where γ is the non-trivial element in Γ . In this situation a 1-cocycle a is basically given by an element of A , say $s := a_\gamma$, satisfying that

$$s\gamma(s) = 1.$$

Let a' be another 1-cocycle, and let $s' := a'_\gamma$. The cocycles a and a' are then cohomologous if there exists $g \in A$ satisfying

$$s' = g^{-1}s\gamma(g).$$

Now, the correspondence (2.4) establishes a bijection between $\text{Conj}(\mathfrak{g})/\sim_c$ and the cohomology set $H^1(\mathbb{Z}/2, \text{Aut}(\mathfrak{g}))$, where here $\mathbb{Z}/2$ is the Galois group of the field extension $\mathbb{R} \subset \mathbb{C}$ acting on the group $\text{Aut}(\mathfrak{g})$ by the rule $\theta \mapsto \tau\theta\tau$.

If \mathfrak{g} is semisimple, a compact real form always exists (see [23, Ch. III], or [37, Sec. 5.1]) and we choose the corresponding conjugation τ as the fixed conjugation in the correspondence given above. Cartan proved that given a conjugation σ of \mathfrak{g} one can choose a conjugation $\sigma' = \alpha\sigma\alpha^{-1}$ where $\alpha \in \text{Int}(\mathfrak{g})$, such that $\sigma'\tau = \tau\sigma'$. Hence $\theta = \sigma'\tau$ is an involution (or equivalently, any Galois cohomology class contains an invariant cocycle).

We define the following equivalence relations for $\theta, \theta' \in \text{Aut}(\mathfrak{g})$:

$$\theta \sim \theta' \text{ if there is } \alpha \in \text{Int}(\mathfrak{g}) \text{ such that } \theta' = \alpha\theta\alpha^{-1},$$

$$\theta \sim_c \theta' \text{ if there is } \alpha \in \text{Aut}(\mathfrak{g}) \text{ such that } \theta' = \alpha\theta\alpha^{-1}.$$

In particular, these define equivalence relations in $\text{Aut}_2(\mathfrak{g})$, the set of involutions of \mathfrak{g} , that is, elements of order 2 in $\text{Aut}(\mathfrak{g})$. Similarly, we define \sim in $\text{Conj}(\mathfrak{g})$ (we have already defined \sim_c above).

We thus have the following.

Proposition 2.1. *The map $\sigma \mapsto \theta := \sigma\tau$ gives bijections*

$$\text{Conj}(\mathfrak{g})/\sim \longleftrightarrow \text{Aut}_2(\mathfrak{g})/\sim,$$

and

$$\text{Conj}(\mathfrak{g})/\sim_c \longleftrightarrow \text{Aut}_2(\mathfrak{g})/\sim_c.$$

In particular, the set of isomorphism classes of real forms of \mathfrak{g} is in one to one correspondence with $\text{Aut}_2(\mathfrak{g})/\sim_c$.

There is yet another equivalence relation for elements $\theta, \theta' \in \text{Aut}_2(\mathfrak{g})$: we say that θ and θ' are **inner equivalent** if their images in $\text{Out}(\mathfrak{g})$ coincide, i.e. if there exists $\alpha \in \text{Int}(\mathfrak{g})$ such that $\theta' = \alpha\theta$.

2.3. Involutions and real forms of a complex Lie group. Let G be a complex Lie group. and let $G_{\mathbb{R}}$ be the underlying real Lie group. We will say that a real Lie subgroup $G_0 \subset G_{\mathbb{R}}$ is a **real form** of G if $G_0 = G^{\sigma}$, the fixed point set of a **conjugation** (antiholomorphic involution) σ of G .

Now, let G be semisimple. A compact real form always exists. From this we can define a conjugation $\tau : G \rightarrow G$ such that $G^{\tau} = U$. This can be seen as follows: The conjugation defining the compact form of \mathfrak{g} can be lifted to a conjugation $\tilde{\tau}$ of the universal cover \tilde{G} . Let $\tilde{U} = \tilde{G}^{\tilde{\tau}}$. Since $Z(\tilde{G})$ is finite, we have $Z(\tilde{G}) \subset \tilde{U}$, and hence $\tilde{\tau}$ acts trivially on $Z(\tilde{G})$ descending to a conjugation τ of G .

We can define for Lie groups the equivalence relations \sim_c and \sim as in the case of Lie algebras.

From Section 2.2, we have bijections

$$\text{Conj}(\tilde{G})/\sim \longleftrightarrow \text{Aut}_2(\tilde{G})/\sim,$$

and

$$\text{Conj}(\tilde{G})/\sim_c \longleftrightarrow \text{Aut}_2(\tilde{G})/\sim_c,$$

where $\text{Aut}_2(\tilde{G})$ is the set of elements of order 2 in $\text{Aut}(\tilde{G})$. One checks that the action on $Z(\tilde{G})$ is the same or the inverse on both sides to conclude the following.

Proposition 2.2. *The map $\sigma \mapsto \theta := \sigma\tau$ gives bijections*

$$\text{Conj}(G)/\sim \longleftrightarrow \text{Aut}_2(G)/\sim,$$

and

$$\text{Conj}(G)/\sim_c \longleftrightarrow \text{Aut}_2(G)/\sim_c,$$

where $\text{Aut}_2(G)$ is the set of elements of order 2 in $\text{Aut}(G)$.

The following is immediate.

Proposition 2.3. (1) *Let $\sigma \sim \sigma' \in \text{Conj}(G)$, say $\sigma' = \text{Int}(g)\sigma\text{Int}(g)^{-1}$ for some $g \in G$, then $G^{\sigma'} = \text{Int}(g)G^{\sigma}$, where G^{σ} and $G^{\sigma'}$ are the subgroups of fixed points of σ and σ' respectively.*

(2) *Similarly, let $\theta, \theta' \in \text{Aut}_2(G)$ such $\theta' = \text{Int}(g)\theta\text{Int}(g)^{-1}$ for some $g \in G$, then $G^{\theta'} = \text{Int}(g)G^{\theta}$, where G^{θ} and $G^{\theta'}$ are the subgroups of fixed points of θ and θ' respectively.*

If G is semisimple then the set of isomorphic real forms of G is in bijection with $\text{Aut}_2(G)/\sim_c$. In particular, the set of equivalence classes of real forms that are inner is given by $\text{Int}_2(G)$, the set of elements of order 2 in $\text{Int}(G)$, modulo conjugation by elements in $\text{Aut}(G)$.

As for Lie algebras, the correspondence given in Proposition 2.2 has again a Galois cohomology interpretation in the sense that $\text{Conj}(G)/\sim_c$ is in bijection with the Galois cohomology group $H^1(\mathbb{Z}/2, \text{Aut}(G))$, where $\mathbb{Z}/2$ is the Galois group of the field extension $\mathbb{R} \subset \mathbb{C}$ acting on the group $\text{Aut}(G)$ by the rule $\theta \mapsto \tau\theta\tau$.

We have the following (see [37, Sec. 3.4]).

Proposition 2.4. *If G is a connected and simply connected Lie group and $s \in \text{Aut}(\mathfrak{g}) (= \text{Aut}(G))$ is a semisimple automorphism, then the group G^s is connected.*

In particular any real form of a connected and simply connected Lie group G is connected.

Remark 2.5. Note that for a complex Lie group G there may exist a real Lie subgroup $G_0 \subset G_{\mathbb{R}}$ whose corresponding Lie algebra \mathfrak{g}_0 is a real form of \mathfrak{g} , without G_0 being a real form of G according to our definition. Examples of this are:

- (1) $G = \mathrm{SL}(2, \mathbb{C})$, $G_0 = \mathrm{NSL}(2, \mathbb{R})$, the normalizer of $\mathrm{SL}(2, \mathbb{R})$ in $\mathrm{SL}(2, \mathbb{C})$;
- (2) $G = \mathrm{Spin}(n, \mathbb{C})$, $G_0 = \mathrm{Spin}(p, q)$ (double cover of $\mathrm{SO}(p, q)$).

These groups G_0 are lifts to G of real forms in $\mathrm{Ad}(G)$.

The set of isomorphism classes of real forms of G is in bijection with $\mathrm{Aut}_2(G)/\sim_c$. Let $[\theta]$ and $[\theta]_c$ denote, respectively, the images of θ under the natural maps $\mathrm{Aut}(G) \rightarrow \mathrm{Aut}(G)/\sim$ and $\mathrm{Aut}(G) \rightarrow \mathrm{Aut}(G)/\sim_c$, respectively. Consider the natural projection $\pi : \mathrm{Aut}(G) \rightarrow \mathrm{Out}(G)$. Let $\theta, \theta' \in \mathrm{Aut}(G)$. Like in the case of Lie algebras, we say that θ and θ' are **inner equivalent** if there is $\alpha \in \mathrm{Int}(G)$ such that $\theta' = \alpha\theta$, or equivalently $\pi(\theta) = \pi(\theta')$.

Proposition 2.6. *Let G be a semisimple complex Lie group.*

- (1) *The projection $\pi : \mathrm{Aut}(G) \rightarrow \mathrm{Out}(G)$ descends to define a surjective map*

$$\mathrm{Aut}(G)/\sim \rightarrow \mathrm{Out}(G),$$

in particular, this defines a surjective map

$$cl : \mathrm{Aut}_2(G)/\sim \rightarrow \mathrm{Out}_2(G),$$

where $\mathrm{Out}_2(G)$ is the set of elements of order 2 in $\mathrm{Out}(G)$ (this is not satisfied if we replace \sim by \sim_c).

- (2) *The action of $\mathrm{Aut}(G)$ by inner automorphisms on $\mathrm{Aut}(G)$ (respectively on $\mathrm{Aut}_2(G)$) induces an action of $\mathrm{Out}(G)$ on $\mathrm{Aut}(G)/\sim$ (respectively $\mathrm{Aut}_2(G)/\sim$) and the quotient is given by $\mathrm{Aut}(G)/\sim_c$ (respectively $\mathrm{Aut}_2(G)/\sim_c$). The map cl is $\mathrm{Out}(G)$ -equivariant. In particular the set $\mathrm{Aut}_2(G)/\sim$ is finite.*

Proof. To prove (1), let $\theta, \theta' \in \mathrm{Aut}(G)$ such that $[\theta] = [\theta']$. This means that $\theta' = \mathrm{Int}(g)\theta \mathrm{Int}(g^{-1})$ for some $g \in G$. One easily checks that $\theta' = \mathrm{Int}(\tilde{g})\theta$, where $\tilde{g} = g\theta(g^{-1})$, thus having $\pi(\theta) = \pi(\theta')$, i.e., θ and θ' are inner equivalent. If θ and θ' are of order two, of course $\pi(\theta)$ and $\pi(\theta')$ are elements in $\mathrm{Out}_2(G)$. It is worth pointing out that, since the extension (2.2) splits, the elements in $\mathrm{Out}_2(G)$ can be lifted to $\mathrm{Aut}_2(G)$.

As for (2), let $\alpha, \theta \in \mathrm{Aut}(G)$. We define the action

$$\alpha \cdot \theta = \alpha\theta\alpha^{-1}.$$

Let $\alpha' = \mathrm{Int}(g)\alpha$. Then

$$\begin{aligned} \alpha' \cdot \theta &= (\mathrm{Int}(g)\alpha) \cdot \theta \\ &= (\mathrm{Int}(g)\alpha)\theta(\mathrm{Int}(g)\alpha)^{-1} \\ &= \mathrm{Int}(g)\alpha \cdot \theta \mathrm{Int}(g)^{-1}, \end{aligned}$$

and hence $[\alpha \cdot \theta] = [\alpha' \cdot \theta]$.

Let $a \in \text{Out}(G)$ and $\alpha \in \pi^{-1}(a) \subset \text{Aut}(G)$. We can thus define the action $a \cdot [\theta] := [\alpha \cdot \theta]$. Of course $[\theta]_c = [\alpha \cdot \theta]_c$, proving the claim. The statement when $\theta \in \text{Aut}_2(G)$ is clear.

Now, Cartan in his classification of real forms for a simple group G shows that $\text{Aut}_2(G)/\sim_c$ is finite. This together with the finiteness of $\text{Out}_2(G)$ (see Table 1) implies that the set $\text{Aut}_2(G)/\sim$ is finite. □

Consider the map $\mathcal{cl} : \text{Aut}_2(G)/\sim \rightarrow \text{Out}_2(G)$ defined in (1) of Proposition 2.6. We will call the image of $[\theta] \in \text{Aut}_2(G)/\sim$ of this map the **clique** of $[\theta]$. Clearly $\mathcal{cl}^{-1}(1) = \text{Int}_2(G)/\sim$, i.e. the $\text{Int}(G)$ -conjugacy classes of inner involutions have trivial clique. Now, if we fix a conjugation $\tau \in \text{Conj}(G)$ defining a compact real form of G , as mentioned in Proposition 2.2, in each class $[\theta] \in \text{Int}_2(G)/\sim$, we can find a representative $\theta = \text{Int}(g)$ in $[\theta]$ for some $g \in G$ such that $\theta\tau = \tau\theta$ and hence $\sigma := \theta\tau$ defines a conjugation inner equivalent to τ , in the sense that $\sigma = \text{Int}(g)\tau$. Real forms σ of G inner equivalent to τ are called **real forms of Hodge type** and the corresponding real groups G^σ are called **groups of Hodge type**.

Combining the bijection

$$\text{Conj}(G)/\sim \longleftrightarrow \text{Aut}_2(G)/\sim,$$

given by Proposition 2.2, with the map $\mathcal{cl} : \text{Aut}_2(G)/\sim \rightarrow \text{Out}_2(G)$, we obtain a map

$$(2.7) \quad \widehat{\mathcal{cl}} : \text{Conj}(G)/\sim \rightarrow \text{Out}_2(G).$$

Of course $\widehat{\mathcal{cl}}^{-1}(1)$ consists of the equivalence classes of real forms of Hodge type.

Since $\text{Int}(G) = \text{Int}(\tilde{G}) = \text{Int}(\mathfrak{g})$, we have in particular $\text{Int}_2(G) = \text{Int}_2(\tilde{G}) = \text{Int}_2(\mathfrak{g})$, and hence every real form of Hodge type on \tilde{G} descends to a real form of Hodge type on G . Of course, any real form on \tilde{G} defines a real form on $\text{Ad}(\tilde{G})$ since the $Z(\tilde{G})$ is invariant under any element in $\text{Conj}(\tilde{G})$. If $G = \tilde{G}/Z'$ for a subgroup $Z' \subset Z(\tilde{G})$, and $\sigma \in \text{Conj}(\tilde{G})$, the condition for σ to define a conjugation on G , and hence a real form on G , is that Z' be invariant under σ .

Proposition 2.7. *Let $\theta \in \text{Aut}_2(G)$. Consider the set*

$$S_\theta := \{s \in G : s\theta(s) = z \in Z(G)\}.$$

Then

- (1) $Z(G)$ acts on S_θ by multiplication.
- (2) G acts on the right on S_θ by

$$s \cdot g := g^{-1}s\theta(g) \quad g \in G, s \in S_\theta.$$

- (3) Let $\pi : \text{Aut}_2(G) \rightarrow \text{Out}_2(G)$ be the natural projection. Let $a \in \text{Out}_2(G)$, and let $\theta \in \pi^{-1}(a)$. Then the map $\psi : S_\theta \rightarrow \pi^{-1}(a)$ defined by $s \mapsto \text{Int}(s)\theta$ gives a bijection

$$S_\theta/(Z(G) \times G) \longleftrightarrow \mathcal{cl}^{-1}(a),$$

where $\mathcal{cl} : \text{Aut}_2(G)/\sim \rightarrow \text{Out}_2(G)$ is the map induced by π .

(4) In particular, let $\theta = \text{Id} \in \text{Aut}_2(G)$. Then

$$S_{\text{Id}} = \{s \in G : s^2 = z \in Z(G)\},$$

and the map $s \mapsto \text{Int}(s)$ defines a bijection

$$S_{\text{Id}}/(Z(G) \times G) \longleftrightarrow \text{Int}_2(G)/\sim.$$

Proof. (1) Let $\theta \in \text{Aut}_2(G)$ and let $s \in S_\theta$. Let $z \in Z(G)$. Since θ leaves $Z(G)$ invariant, we have that $zs\theta(zs) = zs\theta(z)\theta(s) = z\theta(z)s\theta(s) \in Z(G)$ and hence $zs \in S_\theta$.

(2) Let $g \in G$ and $s \in S_\theta$. Let $s \cdot g := g^{-1}s\theta(g)$. We have $(s \cdot g)\theta(s \cdot g) = g^{-1}s\theta(g)\theta(g^{-1})\theta(s)g = g^{-1}s\theta(s)g = s\theta(s)$ and hence $s \cdot g \in S_\theta$.

(3) One first checks that for $g \in G$, the element $\text{Int}(g)\theta$ is in $\text{Aut}_2(G)$ if and only if $g \in S_\theta$. This is clear since, as one can easily compute, $(\text{Int}(g)\theta)^2 = 1$ is equivalent to $g\theta(g)x\theta(g^{-1})g^{-1} = x$ for every $x \in G$, and hence $g\theta(g) \in Z(G)$. We thus have that $s \mapsto \text{Int}(s)\theta$ defines a surjective map $\psi : S_\theta \rightarrow \pi^{-1}(a)$. Obviously, ψ descends to a map on $S_\theta/Z(G)$ since $\text{Int}(s) = \text{Int}(zs)$ for every $z \in Z(G)$.

Let $s' := s \cdot g = g^{-1}s\theta(g)$, where, as above, $s \in S_\theta$ and $g \in G$. Then $\psi(s') = \text{Int}(s')\theta = \text{Int}(g^{-1}s\theta(g))\theta = \text{Int}(g^{-1})\text{Int}(s)\text{Int}(\theta(g))\theta$. But since $\text{Int}(\theta(g))\theta = \theta\text{Int}(g)$, we thus have $\psi(s') = \text{Int}(g^{-1})\psi(s)\text{Int}(g)$, and hence $\psi(s) \sim \psi(s')$.

(4) follows from (3). □

Proposition 2.7 gives indeed an interpretation in terms of non-abelian cohomology (see Section 2.2 for the basic definitions) of $\mathcal{C}l^{-1}(a)$, that is of the set of elements in $\text{Aut}_2(G)/\sim$ with clique a , in particular of the set $\text{Int}_2(G)/\sim$, corresponding to the trivial clique. We have the following.

Proposition 2.8. *Let $H_\theta^1(\mathbb{Z}/2, \text{Ad}(G))$ be the cohomology set defined by the action of $\mathbb{Z}/2$ in $\text{Ad}(G)$ given by $\theta \in \text{Aut}_2(G)$. Then, there is a bijection*

$$H_\theta^1(\mathbb{Z}/2, \text{Ad}(G)) \longleftrightarrow S_\theta/(Z(G) \times G).$$

Proof. Consider the group $\mathbb{Z}/2$ generated by θ . This group is isomorphic to $\mathbb{Z}/2$. Consider now the action of $\mathbb{Z}/2$ on $\text{Ad}(G)$ given by the action of θ (we are denoting the automorphism of $\text{Ad}(G)$ defined by θ also by θ). Let $Z_\theta^1(\mathbb{Z}/2, \text{Ad}(G))$ be the set of cocycles of $\mathbb{Z}/2$ in $\text{Ad}(G)$ given by this action. Let $s \in S_\theta$ and let \tilde{s} the image of s in $\text{Ad}(G)$. The correspondence $s \mapsto \tilde{s}$ defines a bijection $S_\theta/Z(G) \rightarrow Z_\theta^1(\mathbb{Z}/2, \text{Ad}(G))$, where here we are identifying a cocycle a with the corresponding element $a_\theta \in \text{Ad}(G)$. The action of G on $S_\theta/Z(G)$ is via the action of $\text{Ad}(G)$, in other words, $S_\theta/(Z(G) \times G) = (S_\theta/(Z(G)))/\text{Ad}(G)$, where $\tilde{g} \in \text{Ad}(G)$ acts on $\tilde{s} \in S_\theta/(Z(G))$ by the rule

$$\tilde{s} \cdot \tilde{g} = \tilde{g}^{-1}\tilde{s}\theta(\tilde{g}).$$

But, this is precisely the condition given in 2.6 for the cocycles a and a' corresponding to \tilde{s} and \tilde{s}' , respectively, to be cohomologous. □

In view of Propositions 2.7 and 2.8, it is clear that if $\theta, \theta' \in \text{Aut}_2(G)$ are such that $\pi(\theta) = \pi(\theta') = a$ then there is a bijection between $H_\theta^1(\mathbb{Z}/2, \text{Ad}(G))$ and $H_{\theta'}^1(\mathbb{Z}/2, \text{Ad}(G))$.

In this sense this cohomology set could very well be denoted by $H_a^1(\mathbb{Z}/2, \text{Ad}(G))$. With this notation, we have shown the following.

Proposition 2.9. *Let $a \in \text{Out}_2(G)$. There is a bijection*

$$c\ell^{-1}(a) \longleftrightarrow H_a^1(\mathbb{Z}/2, \text{Ad}(G)),$$

and hence a bijection

$$\text{Aut}_2(G)/\sim \longleftrightarrow \bigcup_{a \in \text{Out}_2(G)} H_a^1(\mathbb{Z}/2, \text{Ad}(G)).$$

So, while the cohomology set $H^1(\mathbb{Z}/2, \text{Aut}(G))$ describes the set of conjugations of G , modulo the equivalence given by \sim_c , $H_\theta^1(\mathbb{Z}/2, \text{Ad}(G))$ is in bijection with the set of equivalence classes of conjugations of G given by the relation \sim (note that $\text{Ad}(G) \cong \text{Int}(G)$). For similar discussion on cohomology and real forms look at the recent paper by J. Adams [1].

2.4. Normalizers, real forms and the isotropy representation. Let G be a complex semisimple Lie group and let $Z := Z(G)$ be the centre of G . Let $\sigma \in \text{Conj}(G)$. We have considered the real form defined by σ , that is a real subgroup of G defined by

$$G^\sigma := \{g \in G : \sigma(g) = g\}.$$

We shall consider now the real subgroup of G defined by

$$G_\sigma := \{g \in G : \sigma(g) = c(g)g, \text{ with } c(g) \in Z\}.$$

Proposition 2.10. *The group G_σ is $N_G(G^\sigma)$, the normalizer of G^σ in G .*

Proof. It is immediate that $G_\sigma \subset N_G(G^\sigma)$.

To show the converse, let $x \in N_G(G^\sigma)$. In order for x to be in G_σ , we need to have that $x^{-1}\sigma(x) \in Z$. For this, it is enough to show that $x^{-1}\sigma(x)$ commutes with every element of G^σ , since the centralizer of $x^{-1}\sigma(x)$ is a complex algebraic group, and if it contains G^σ it must be the whole G . Indeed, if $g \in G^\sigma$, then $xgx^{-1} \in G^\sigma$ since $x \in N_G(G^\sigma)$, and hence $\sigma(xgx^{-1}) = xgx^{-1}$. But $\sigma(xgx^{-1}) = \sigma(x)g\sigma(x^{-1})$. These two equalities combined imply that $x^{-1}\sigma(x)$ commutes with g as we wanted to show. \square

The conjugation σ leaves Z invariant and hence descends to a conjugation of $\text{Ad}(G) = G/Z$. The following is immediate.

Proposition 2.11. $\text{Ad}(G)^\sigma = \text{Ad}(G_\sigma) := G_\sigma/Z(G_\sigma)$, where we denote also by σ the conjugation defined on $\text{Ad}(G)$. In other words, G_σ is a lift to G of the real form in $\text{Ad}(G)$ defined by σ .

Define $\Gamma_\sigma = G_\sigma/G^\sigma$. We have the exact sequence

$$(2.8) \quad 1 \longrightarrow G^\sigma \longrightarrow G_\sigma \longrightarrow \Gamma_\sigma \longrightarrow 1.$$

Similarly, if $\theta \in \text{Aut}_2(G)$ we can define the subgroups

$$G^\theta := \{g \in G : \theta(g) = g\},$$

$$G_\theta := \{g \in G : \theta(g) = c(g)g, \text{ with } c(g) \in Z\}.$$

As in the case of conjugations, an element $\theta \in \text{Aut}_2(G)$ leaves Z invariant and hence descends to an automorphism of order 2 of $\text{Ad}(G)$. We have the following.

Proposition 2.12. $\text{Ad}(G)^\theta = \text{Ad}(G_\theta) := G_\theta/Z(G_\theta)$, where we denote also by θ the involution defined on $\text{Ad}(G)$.

It is clear that $Z \subset G_\theta$. For any $g \in G$ we have $\theta(g) = \theta(g)g^{-1}g$. But if $g \in Z$, $\theta(g)$ is also in the centre and so is $\theta(g)g^{-1}$. Hence $g \in G_\theta$ with $c(g) = \theta(g)g^{-1}$.

Remark 2.13. We thus have that G_θ contains $G^\theta Z$, but may be larger. Take θ corresponding to the conjugation defining $\text{SL}(2, \mathbb{R})$. In this case the group G_θ contains the diagonal matrix with entries $(i, -i)$, and is in fact the normalizer of $\text{SO}(2, \mathbb{C})$ in $\text{SL}(2, \mathbb{C})$.

Remark 2.14. Note that if θ is an inner involution i.e. $\theta = \text{Int}(g)$ for some $g \in G$ (real forms of Hodge type), then $Z \subset G^\theta$.

Again G_θ normalizes G^θ and we have an exact sequence

$$(2.9) \quad 1 \longrightarrow G^\theta \longrightarrow G_\theta \longrightarrow \Gamma_\theta \longrightarrow 1.$$

Proposition 2.15. (1) The map $c : G_\theta \longrightarrow Z$ defined by $c(g)$ appearing in the definition of G_θ , is a homomorphism. We have $\ker c = G^\theta$ and hence c induces an injective homomorphism $\tilde{c} : \Gamma_\theta \longrightarrow Z$.

(2) The action of θ on G restricts to an action on G_θ , and c is θ -equivariant with respect to this action and the natural action of θ on Z . Moreover this action descends to Γ_θ and hence \tilde{c} is θ -equivariant.

(3) $\theta(c(g)) = c(g^{-1})$ and hence the image of \tilde{c} in Z is contained in

$$Z_a := \{z \in Z : a(z) = z^{-1}\},$$

and contains

$$\{a(z)z^{-1} : z \in Z\},$$

with $a = \pi(\theta)$, where π is the projection $\text{Aut}(G) \rightarrow \text{Out}(G)$.

Proof. Let $g_1, g_2 \in G_\theta$, we have

$$\theta(g_1 g_2) = \theta(g_1) \theta(g_2) = c(g_1) g_1 c(g_2) g_2 = c(g_1) c(g_2) g_1 g_2,$$

which implies that $c(g_1 g_2) = c(g_1) c(g_2)$, proving the first statement in (1).

Clearly, $\ker c = G^\theta$, and hence we complete the proof of (1).

To prove (2) we have that if $g \in G_\theta$ then $\theta(g) = c(g)g$ with $c(g) \in Z$. Now $\theta(c(g)g) = \theta(c(g))\theta(g) = \theta(c(g))c(g)g$. Hence, since $\theta(c(g)) \in Z$, we conclude that $c(g)g \in G_\theta$, showing that θ does indeed act on G_θ . Our computation also shows that $c(\theta(g)) = \theta(c(g))$ and hence c is θ -equivariant. The action of θ on G_θ fixes G^θ and hence induces an action on Γ_θ , giving that \tilde{c} is a θ -equivariant injective homomorphism.

(3) Now, $g = \theta^2(g) = \theta(c(g)g) = \theta(c(g))\theta(g) = \theta(c(g))c(g)g$, from which we conclude that $\theta(c(g)) = c(g)^{-1}$. Since the action of θ on Z depends only on the clique $a = \pi(\theta)$, the remaining now follows from (2). \square

Proposition 2.16. *Let $\theta, \theta' \in \text{Aut}_2(G)$ such that $\theta \sim \theta'$ with $\theta' = \text{Int}(g)\theta \text{Int}(g)^{-1}$ for $g \in G$.*

(1) *The map $x \mapsto \text{Int}(g)x$ defines an isomorphism $f_g : G_\theta \rightarrow G_{\theta'}$, which induces an isomorphism $\tilde{f}_g : \Gamma_\theta \rightarrow \Gamma_{\theta'}$.*

(2) *Let $c : G_\theta \rightarrow Z$ and $c' : G_{\theta'} \rightarrow Z$ be the homomorphisms corresponding to θ and θ' , and $\tilde{c} : \Gamma_\theta \rightarrow Z$ and $\tilde{c}' : \Gamma_{\theta'} \rightarrow Z$ be the induced homomorphisms as defined in (1) of Proposition 2.15. Then $c = c'f_g$ and $\tilde{c} = \tilde{c}'\tilde{f}_g$,*

Proof. (1) The first statement is immediate. The second follows from this and (2) in Proposition 2.3.

To prove (2), let $x \in G_\theta$, and let $x' = f_g(x) = \text{Int}(g)(x)$. Then $c'(x')x' = \theta'(x') = \text{Int}(g)\theta \text{Int}(g)^{-1}(x') = \text{Int}(g)\theta(x) = \text{Int}(g)(c(x)x) = c(x)\text{Int}(g)(x) = c(\text{Int}(g)^{-1}(x'))x'$. The result follows. \square

Let σ be a conjugation of G and choose a compact conjugation τ of G such that $\tau\sigma = \sigma\tau =: \theta$. The group $U = G^\tau$ is a maximal compact subgroup of G . Since σ and τ commute, σ acts on U and $U^\sigma = U \cap G^\sigma$ is a maximal compact subgroup of G^σ . We can also consider

$$U_\sigma := \{u \in U : \sigma(u) = c(u)u, \text{ with } c(u) \in Z(U) = Z\}.$$

We have that $U_\sigma = U \cap G_\sigma$ is a maximal compact subgroup of G_σ . We thus have the exact sequence

$$(2.10) \quad 1 \longrightarrow U^\sigma \longrightarrow U_\sigma \longrightarrow \Gamma_\sigma \longrightarrow 1.$$

By complexifying this, comparing with (2.9), and the fact that Γ_θ is a finite group (this is clear if G is semisimple since Z is finite). We have proved the following.

Proposition 2.17. *Let σ be a conjugation of G and τ a compact conjugation of G commuting with σ . Let $\theta = \sigma\tau$. Then*

$$\Gamma_\theta = \Gamma_\sigma.$$

Proposition 2.18. *The group U_σ is $N_U(U^\sigma)$, the normalizer of U^σ in U .*

Proof. The proof follows from Proposition 2.10 and the fact that σ commutes with τ , and $U^\sigma \subset G^\sigma$ and $U_\sigma \subset G_\sigma$ are maximal compact subgroups. \square

Proposition 2.19. *The group G_θ is $N_G(G^\theta)$, the normalizer of G^θ in G .*

Proof. Follows from Proposition 2.18 and the fact that G , G^θ and G_θ are the complexifications U , U^σ and U_σ , respectively. \square

Remark 2.20. If $\theta, \theta' \in \text{Aut}_2(G)$ are in the same clique, i.e. $\pi(\theta) = \pi(\theta')$, but $[\theta] \neq [\theta']$, then Γ_θ and $\Gamma_{\theta'}$ need not be isomorphic. An example is provided by $G = \text{SL}(2n, \mathbb{C})$, θ corresponding to $\text{SU}(n, n)$ and θ' corresponding to $\text{SU}(p, q)$ with $p \neq q$. In this situation, the normalizer of $\text{SU}(n, n)$ in $\text{SL}(n, \mathbb{C})$ has two connected components, while the normalizer of $\text{SU}(p, q)$ coincides with $\text{SU}(p, q)$.

An element $\theta \in \text{Aut}_2(G)$ defines a Cartan decomposition of \mathfrak{g} in (± 1) -eigenspaces:

$$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-,$$

satisfying $[\mathfrak{g}^+, \mathfrak{g}^+] \subset \mathfrak{g}^+$, $[\mathfrak{g}^-, \mathfrak{g}^-] \subset \mathfrak{g}^+$, and $[\mathfrak{g}^+, \mathfrak{g}^-] \subset \mathfrak{g}^-$. Clearly \mathfrak{g}^+ is the Lie algebra of G^θ . We have the following.

Proposition 2.21. (1) *The restriction of the adjoint representation of G to G^θ defines representations*

$$\iota^\pm : G^\theta \rightarrow \text{GL}(\mathfrak{g}^\pm).$$

(2) *The restriction of the adjoint representation of G to G_θ defines representations*

$$\iota_\pm : G_\theta \rightarrow \text{GL}(\mathfrak{g}^\pm).$$

Proof. (1) is well known: The restriction of the adjoint representation of G to G^θ gives indeed the adjoint representation of G^θ in \mathfrak{g}^+ and the **isotropy** representation in \mathfrak{g}^- (see e.g. [23]).

(2) is a consequence of (1) applied to $\text{Ad}(G)$, together with Proposition 2.12, and the facts that the Lie algebras and Cartan decompositions for G and $\text{Ad}(G)$ under θ coincide, and the adjoint representation of G factors through the adjoint representation of $\text{Ad}(G)$. \square

2.5. Finite order automorphisms of G . Let G be a complex semisimple Lie group. We show now how many results of Section 2.3 generalise to automorphisms of G of arbitrary finite order.

Let $\text{Aut}_n(G)$ and $\text{Out}_n(G)$ be the set of elements of order n in $\text{Aut}(G)$ and $\text{Out}(G)$, respectively. A straightforward generalisation of Proposition 2.7 is the following (we leave the proof to the reader).

Proposition 2.22. *Let $\theta \in \text{Aut}_n(G)$. Consider the set*

$$S_\theta^n := \{s \in G : s\theta(s) \cdots \theta^{n-1}(s) = z \in Z(G)\}.$$

Then

(1) *$Z(G)$ acts on S_θ^n by multiplication.*

(2) *G acts on the right on S_θ^n by*

$$s \cdot g := g^{-1}s\theta(g) = g \in G, s \in S_\theta^n.$$

(3) *Let $\pi : \text{Aut}_n(G) \rightarrow \text{Out}_n(G)$ be the natural projection. Let $a \in \text{Out}_n(G)$, and let $\theta \in \pi^{-1}(a)$. Then the map $\psi : S_\theta^n \rightarrow \pi^{-1}(a)$ defined by $s \mapsto \text{Int}(s)\theta$ gives a bijection*

$$S_\theta^n / (Z(G) \times G) \longleftrightarrow \mathcal{C}_n^{-1}(a),$$

where $\mathcal{C}_n : \text{Aut}_n(G)/\sim \rightarrow \text{Out}_n(G)$ is the map induced by π .

(4) In particular, let $\theta = \text{Id} \in \text{Aut}_n(G)$. Then

$$S^n := S_{\text{Id}}^n = \{s \in G : s^n = z \in Z(G)\},$$

and the map $s \mapsto \text{Int}(s)$ defines a bijection

$$S^n/(Z(G) \times G) \longleftrightarrow \text{Int}_n(G)/\sim.$$

We will refer to an element of $\text{Out}_n(G)$ as an n -**clique**. We also have an interpretation of $\mathcal{C}_n^{-1}(a)$ in terms of non abelian cohomology given by the following generalisation of Proposition 2.8.

Proposition 2.23. *Let $\theta \in \text{Aut}_n(G)$. Consider the natural action of the group generated by θ , which is isomorphic to \mathbb{Z}/n , on $\text{Ad}(G)$. Let $H_\theta^1(\mathbb{Z}/n, \text{Ad}(G))$ be the first cohomology set defined by this action. Let S_θ^n be as defined in Proposition 2.22 with the action of $Z(G) \times G$ on it defined in that proposition. Then, there is a bijection*

$$H_\theta^1(\mathbb{Z}/n, \text{Ad}(G)) \longleftrightarrow S_\theta^n/(Z(G) \times G).$$

Proof. The proof is indeed very similar to that of Proposition 2.23. Let $Z_\theta^1(\mathbb{Z}/n, \text{Ad}(G))$ be the set of cocycles of \mathbb{Z}/n in $\text{Ad}(G)$ with the given action. Let $s \in S_\theta^n$ and let \tilde{s} the image of s in $\text{Ad}(G)$. The correspondence $s \mapsto \tilde{s}$ defines a bijection $S_\theta^n/Z(G) \rightarrow Z_\theta^1(\mathbb{Z}/n, \text{Ad}(G))$, where here we are identifying a cocycle a with the corresponding element $a_\theta \in \text{Ad}(G)$, since the elements a_{θ^i} for $1 \leq i \leq n$ are determined by the recursive formula

$$a_{\theta^i} = a_\theta \theta(a_\theta^{i-1})$$

given by the cocycle condition (2.5). Clearly if $a_\theta := \tilde{s}$, then

$$1 = a_{\theta^n} = \tilde{s} \theta(\tilde{s}) \cdots \theta^{n-1}(\tilde{s}),$$

proving that $s \in S_\theta^n$. According to (2.6) two cocycles are cohomologous if there is $\tilde{g} \in \text{Ad}(G)$ such that $a'_\theta = \tilde{g}^{-1} a_\theta \theta(\tilde{g})$ which coincides with the action of $\text{Ad}(G)$ on $S_\theta^n/Z(G)$. \square

Similarly to Proposition 2.9, we have the following.

Proposition 2.24. *Let $a \in \text{Out}_n(G)$. There is a bijection*

$$\mathcal{C}_n^{-1}(a) \longleftrightarrow H_a^1(\mathbb{Z}/n, \text{Ad}(G)),$$

and hence a bijection

$$\text{Aut}_n(G)/\sim \longleftrightarrow \bigcup_{a \in \text{Out}_n(G)} H_a^1(\mathbb{Z}/n, \text{Ad}(G)).$$

One way of understanding the ‘twisting’ defined by $\theta \in \text{Aut}_n(G)$ is provided by the following.

Proposition 2.25. *Let $\theta \in \text{Aut}_n(G)$ and $\widehat{G} := G \rtimes \mathbb{Z}/n$ be the semidirect product defined by the natural action of $\mathbb{Z}/n = \langle \theta \rangle$ on G . We have the following:*

- (1) $s \in S_\theta^n$ if and only if $(s, \theta)^n \in Z(G)$ (where $Z(G)$ here is identified with $Z \times \{1\} \subset \widehat{G}$).
- (2) Let $\widehat{S}^n = \{\hat{s} \in \widehat{G} : \hat{s}^n \in Z(G)\}$. Then $S_\theta^n/G = \widehat{S}^n/\widehat{G}$, where \widehat{G} acts on \widehat{S}^n by conjugation.

Proof. Let $*$ denote the group operation in \widehat{G} . Then if $s, s' \in G$, we have $(s, \theta^i) * (s', \theta^j) = (s\theta^i(s'), \theta^{i+j})$ and hence, by induction,

$$(s, \theta)^n = (s\theta(s) \cdots \theta^{n-1}(s), 1).$$

We thus conclude that $s \in S_\theta^n$ if and only if $(s, \theta)^n \in Z(G)$, and (1) follows.

To prove (2) we embed G as the subset $G' \subset \widehat{G}$ by $s \mapsto (\theta(s), \theta)$. Clearly if $s \in S_\theta^n$, $\theta(s) \in S_\theta^n$ and, from (1), this is equivalent to $(\theta(s), \theta)^n \in Z(G)$. Moreover, the conjugacy action of \widehat{G} leaves G' invariant and when restricted to the subgroup G gives the action by $g \in G$ us $s \mapsto g^{-1}\theta(s)g$.

□

As for involutions, for $\theta \in \text{Aut}_n(G)$, we can define the subgroup

$$G_\theta := \{g \in G : \theta(g) = c(g)g, \text{ with } c(g) \in Z\}.$$

Again, like in the $n = 2$ case, G_θ normalizes G^θ and we have an exact sequence

$$(2.11) \quad 1 \longrightarrow G^\theta \longrightarrow G_\theta \longrightarrow \Gamma_\theta \longrightarrow 1.$$

Propositions 2.12 and 2.15 generalise immediately to the following.

Proposition 2.26. (1) $\text{Ad}(G)^\theta = \text{Ad}(G_\theta) := G_\theta/Z(G_\theta)$, where we denote also by θ the involution defined on $\text{Ad}(G)$.

(2) The map $c : G_\theta \longrightarrow Z$ defined by $c(g)$ for $c(g)$ appearing in the definition of G_θ is a homomorphism, with $\ker c = G^\theta$, and hence inducing an injective homomorphism $\tilde{c} : \Gamma_\theta \longrightarrow Z$.

(3) The action of θ on G restricts to an action on G_θ , and c is θ -equivariant with respect to this action and the natural action of θ on Z . Moreover this action descends to Γ_θ and hence \tilde{c} is θ -equivariant.

(4) For every $g \in G_\theta$, $c(g)$ satisfies the equation

$$c(g)\theta(c(g)) \cdots \theta^{n-1}(c(g)) = 1$$

and hence the image of \tilde{c} in Z is contained in

$$Z_a := \{z \in Z : za(z) \cdots a^{n-1}(z) = 1\},$$

where $a = \pi(\theta)$.

We can decompose \mathfrak{g} as

$$\mathfrak{g} = \bigoplus_{k=0}^{n-1} \mathfrak{g}^k,$$

where \mathfrak{g}^k is the eigenspace of the automorphism of \mathfrak{g} defined by θ with eigenvalue $\zeta_k := \exp(2\pi i \frac{k}{n})$.

Clearly \mathfrak{g}^0 is the Lie algebra of $G^\theta := \{g \in G : \theta(g) = g\}$. We also have $[\mathfrak{g}^l, \mathfrak{g}^k] \subset \mathfrak{g}^{l+k}$, in particular $[\mathfrak{g}^0, \mathfrak{g}^k] \subset \mathfrak{g}^k$, which lift to actions of G^θ and G_θ on \mathfrak{g}^k . More precisely:

Proposition 2.27. (1) *The restriction of the adjoint representation of G to G^θ defines representations*

$$\iota^k : G^\theta \rightarrow \mathrm{GL}(\mathfrak{g}^k).$$

(2) *The restriction of the adjoint representation of G to G_θ defines representations*

$$\iota_k : G_\theta \rightarrow \mathrm{GL}(\mathfrak{g}^k).$$

2.6. Groups of Hermitian type. In this section G is a complex simple Lie group. The semisimple case follows from this. We use the notation of Sections 2.3 and 2.4. Let $\theta \in \mathrm{Aut}_2(G)$, and τ be a compact conjugation of G commuting with θ , defining the conjugation of G given by $\sigma = \tau\theta$. Let G^θ , G_θ , G^σ , G_σ , U , U^σ , and U_σ be as in Section 2.4.

We consider in this section a particular class of conjugations of G (and hence involutions) which define what are called groups of **Hermitian type**. These are conjugations σ for which the symmetric space $M_\sigma := G^\sigma/U^\sigma$ is of Hermitian type, in fact a Kähler manifold [28, 23]. These are distinguished by the fact that $\mathfrak{z}(\mathfrak{u}^\sigma)$, the centre of \mathfrak{u}^σ , is isomorphic to \mathbb{C} . A base element in $\mathfrak{z}(\mathfrak{u}^\sigma)$ defines via de adjoint representation a complex structure in \mathfrak{g}^- , the tangent space of M_σ at the point corresponding to the coset U^σ . This applies also to G_σ . In fact the symmetric space defined by G_σ , given by G_σ/U_σ coincides with M_σ . It is well-known that groups of Hermitian type are of Hodge type, as defined in Section 2.3 (see [23]).

The conjugations of G that are not of Hermitian type have the property that $\mathfrak{z}(\mathfrak{u}^\sigma) = 0$ (see [23]), and hence the groups U^σ , G^θ , U_σ , and G_θ are semisimple. This difference between Hermitian and non-Hermitian groups will have important consequences in the theory of G -Higgs bundles studied below.

3. FINITE ORDER AUTOMORPHISMS OF PRINCIPAL BUNDLES

In this section G is a complex semisimple Lie group and E is a holomorphic principal G -bundle over a compact Riemann surface X .

The centre $Z := Z(G)$ of G acts on E by $\xi \rightarrow \xi z$, for every $\xi \in E$ and $z \in Z$. This gives an inclusion $Z \subset \mathrm{Aut}(E)$ as a subgroup. Moreover, Z is in the centre of $\mathrm{Aut}(E)$, for if $z \in Z$ and $A \in \mathrm{Aut}(E)$ then $Az(\xi) = A(\xi z) = A(\xi)z = z(A(\xi))$.

In this section we study how an element $A \in \mathrm{Aut}(E)$ such that $A^n = z \in Z$ for some n gives rise to a reduction of structure group of the bundle E . Such an A defines an automorphism of finite order of the $\mathrm{Ad}(G)$ -bundle associated to E . We will generalise this to ‘twisted’ automorphisms in a sense that we will explain.

3.1. Finite order automorphisms and reductions of structure group. Let E be a principal bundle and A an automorphism. Then there is a natural morphism $f_A : E \rightarrow G$ given by $A(\xi) = \xi f_A(\xi)$, for all $\xi \in E$.

Lemma 3.1. *The map f_A is equivariant for the right action of G on E and the (right) adjoint action of G on itself namely $x \rightarrow g^{-1}xg$ for $x, g \in G$.*

Proof. In fact, if $g \in G, \xi \in E$, we have $A(\xi g) = (\xi g)f_A(\xi g)$ on the one hand, and $A(\xi)g = \xi f_A(\xi)g$ on the other, leading to the equality $gf_A(\xi g) = f_A(\xi)g$. \square

Lemma 3.2. *If $A_1, A_2 \in \text{Aut}(E)$, then $f_{A_1 A_2} = f_{A_1} f_{A_2}$.*

Proof. In fact, $\xi f_{A_1 A_2}(\xi) = (A_1 A_2)(\xi) = A_1(A_2 \xi) = A_1(\xi f_{A_2}(\xi)) = A_1(\xi) f_{A_2}(\xi) = \xi f_{A_1}(\xi) f_{A_2}(\xi)$, proving our assertion. \square

Proposition 3.3. *Let $A \in \text{Aut}(E)$ be such that $A^n = z \in Z$. Then:*

(1) *f_A maps E onto a single orbit $S(E)$ of the set*

$$S^n := \{s \in G : s^n = z \in Z\}$$

under the right action of G by inner automorphisms.

(2) *Every element $s \in S(E)$ defines a reduction of structure group of E to $Z_G(s)$.*

(3) *The automorphism A leaves the $Z_G(s)$ -bundle invariant and coincides there with the central element s .*

Proof. By Lemma 3.2, $f_{A^n} = (f_A)^n$ is the constant map $\xi \rightarrow z$ for an element $z \in Z(G)$. Hence f_A maps E into S^n . The morphism f_A defines a morphism $\tilde{f}_A : E \rightarrow G // G$, where $G // G$ is the GIT quotient for the (right) action of G on itself given by $x \mapsto g^{-1}xg$ for $x, g \in G$. Since \tilde{f}_A is constant on the fibres (Lemma 3.1), it descends to a morphism $X = E/G \rightarrow G // G$, which must be constant since X is projective and $G // G$ is affine. But $f_A(\xi) \in S^n$ and hence is a semisimple element of G for all $\xi \in E$ and hence a stable point for the GIT quotient $G // G$. Therefore $f_A(\xi)$ for every $\xi \in E$ lie on a single orbit for the given action of G on S^n as claimed.

To prove (2), let $s \in S(E)$, and $S := f_A^{-1}(s)$. Clearly ξ and ξ' on a fibre of $E \rightarrow X$ belong to S if and only if $\xi' = \xi g$ for some $g \in G$, and $f_A(\xi) = s$ and $s = f_A(\xi') = f_A(\xi g) = g^{-1}f_A(\xi)g = g^{-1}sg$. This implies that $g \in Z_G(s)$. Thus the set S is acted upon transitively on fibres by $Z_G(s)$ giving an $Z_G(s)$ -bundle to which E is reduced.

Assertion (3) is obvious. \square

3.2. Twisted automorphisms of principal bundles. Let E be a G -bundle over X . Let $\theta \in \text{Aut}(G)$. We define the set of θ -**twisted automorphisms** of E by

$$\text{Aut}_\theta(E) := \{A : E \rightarrow E \text{ bijective} : A(\xi g) = A(\xi)\theta(g) \text{ for } \xi \in E, g \in G\}.$$

Twisted automorphisms are related to ordinary isomorphisms between E and a related bundle. More precisely, the following is immediate.

Proposition 3.4. *Let E be a G -bundle over X . Let $\theta \in \text{Aut}(G)$. Then:*

(1) *The G -bundle $\theta(E)$ is isomorphic to the G -bundle whose total space is E with a G -action defined by*

$$e \cdot g := e\theta(g) \text{ for } e \in E, g \in G.$$

(2) *Under the isomorphism given in (1) an isomorphism $E \rightarrow \theta(E)$ can be identified with a θ -twisted automorphism of E .*

Let $A \in \text{Aut}_\theta(E)$. We define the function $f_A : E \rightarrow G$ by the formula

$$(3.1) \quad A(\xi) = \xi f_A(\xi) \text{ for every } \xi \in E.$$

Lemma 3.5. *Let $A \in \text{Aut}_\theta(E)$ and let $f_A : E \rightarrow G$ the function given by (3.1). The function f_A is G -equivariant for the right action of G on E and the right action of G on itself given by*

$$x \mapsto g^{-1}x\theta(g) \text{ for } x, g \in G.$$

Proof. Let $\xi \in E$ and $g \in G$. On the one hand we have $A(\xi g) = (\xi g)f_A(\xi g)$. On the other hand $A(\xi g) = A(\xi)\theta(g) = \xi f_A(\xi)\theta(g) = \xi g g^{-1} f_A(\xi)\theta(g)$, thus concluding that $f_A(\xi g) = g^{-1}f_A(\xi)\theta(g)$. \square

Lemma 3.6. *Let $\theta_1, \theta_2 \in \text{Aut}(G)$, and $A_1 \in \text{Aut}_{\theta_1}(E)$ and $A_2 \in \text{Aut}_{\theta_2}(E)$. Then*

$$(1) \quad A_1 A_2 \in \text{Aut}_{\theta_1 \theta_2}(E).$$

(2) $f_{A_1 A_2} = f_{A_1} \cdot \theta_1(f_{A_2})$, where this means that $f_{A_1 A_2}(\xi) = f_{A_1}(\xi)\theta_1((f_{A_2})(\xi))$, for every $\xi \in E$.

Proof. Let $\xi \in E$ and $g \in G$.

$$\begin{aligned} A_1(A_2(\xi g)) &= A_1(A_2(\xi)\theta_2(g)) \\ &= A_1(A_2(\xi))\theta_1(\theta_2(g)) \\ &= (A_1 A_2)(\xi)(\theta_1 \theta_2)(g), \end{aligned}$$

proving (1). The proof of (2) is given by the following computation:

$$\begin{aligned} \xi f_{A_1 A_2}(\xi) &= (A_1 A_2)(\xi) \\ &= A_1(A_2(\xi)) \\ &= A_1(\xi f_{A_2}(\xi)) \\ &= A_1(\xi)\theta_1(f_{A_2}(\xi)) \\ &= \xi f_{A_1}(\xi)\theta_1(f_{A_2}(\xi)). \end{aligned}$$

\square

From Lemma 3.6 we conclude the following.

Proposition 3.7. *Let $\theta \in \text{Aut}_n(G)$ and let E be a G -bundle. The set*

$$\widehat{\text{Aut}(E)} := \bigcup_{i=0}^{n-1} \text{Aut}_{\theta^i}(E)$$

is a (possibly disconnected) group fitting in an exact sequence

$$1 \rightarrow \text{Aut}(E) \rightarrow \widehat{\text{Aut}(E)} \rightarrow \mathbb{Z}/n.$$

One can easily prove the following.

Proposition 3.8. *Let E be a G -bundle and \widehat{E} be the \widehat{G} -bundle associated to E by extension of structure group, where \widehat{G} is the group defined in Proposition 2.25. Then*

$$\widehat{\text{Aut}(E)} \cong \text{Aut}(\widehat{E}).$$

Here is the twisted version of Proposition 3.3.

Proposition 3.9. *Let E be a G -bundle over a compact Riemann surface X . Let $\theta \in \text{Aut}_n(G)$ and $A \in \text{Aut}_\theta(E)$ such that $A^n = z \in Z(G) \subset \text{Aut}(E)$. Then:*

(1) *The function f_A given in (3.1) maps E onto a single orbit $S(E)$ of the set $S_\theta^n := \{s \in G : s\theta(s) \cdots \theta^{n-1}(s) = z \in Z(G)\}$ defined in Proposition 2.22 under the right action of G defined there, namely $s \cdot g = g^{-1}s\theta(g)$.*

(2) *Every element $s \in S(E)$ defines a reduction of structure group of E to $G^{\theta'}$, where $\theta' = \text{Int}(s)\theta$, and $G^{\theta'}$ is the subgroup of G of fixed points under θ' .*

Proof. We adapt the proof of Proposition 3.3 to the twisted situation. By Lemma 3.6, $f_{A^n} = f_A \cdot \theta(f_A) \cdots \theta^{n-1}(f_A) = f_z$ where $f_z : E \rightarrow G$ is the constant map given by $f_z(\xi) = z$ for an element $z \in Z(G)$ and every $\xi \in E$. By setting $s := f_A(\xi)$, this implies that $s \in S_\theta^n$.

The morphism f_A defines a morphism $\tilde{f}_A : E \rightarrow G //_\theta G$, where $G //_\theta G$ is the GIT quotient for the action of G on itself given by the θ -twisted action $x \mapsto g^{-1}x\theta(g)$ for $x, g \in G$. Since \tilde{f}_A is constant on the fibres (Lemma 3.5), it descends to a morphism $X = E/G \rightarrow G //_\theta G$, which must be constant since X is projective and $G //_\theta G$ is affine.

Now, we claim that every element $s \in S_\theta^n$ is stable for the θ -twisted action. This follows from Proposition 2.25, since the element $s \in S_\theta^n$ is in correspondence with an element of order n in the twisted group \widehat{G} defined in Proposition 2.25 and is hence a semisimple element of \widehat{G} . We thus deduce that $f_A(\xi) \in S_\theta^n$ is a stable point for the GIT quotient $G //_\theta G$, and hence $f_A(\xi)$ for every ξ lie on a single orbit for the θ -twisted action of G on S_θ^n as we claim in (1).

The proof of (2) is a straightforward generalisation of that of (2) in Proposition 3.3. □

One may consider a different kind of automorphisms of a G -bundle that involves a central subgroup Γ (in particular $\Gamma = Z$). Given $\theta \in \text{Aut}(G)$, we define a **(θ, Γ) -twisted automorphism** to be a fibre-preserving morphism $A : E \rightarrow E$ satisfying $A(\xi g) = A(\xi)z\theta(g)$ for all $\xi \in E$ and $g \in G$ with $z \in \Gamma$ (depending on A, θ, ξ and g). We will denote the set of (θ, Γ) -twisted automorphisms of E by $\text{Aut}_\theta^\Gamma(E)$.

Let then E (resp. α) be a principal G -bundle (resp. Γ -bundle). The fibre product $E \times_X \alpha$ is then in a natural way a principal $(G \times \Gamma)$ -bundle. Since Γ is abelian, we may, and will often, write the action of Γ on α on the left.

Using the homomorphism $m : G \times \Gamma \rightarrow G$ given by multiplication, we get by extension of structure group, a principal G -bundle which we denote by $E \otimes \alpha$. This is clearly a quotient of $E \times_X \alpha$ by the action of Γ given by $z(\xi, a) = (\xi z^{-1}, za)$. We will denote the image of (ξ, a) in $E \otimes \alpha$ by $\xi \otimes a$. Notice that for any $z \in \Gamma$ we have $\xi \cdot z \otimes a = \xi \otimes za$ for all $a \in \Gamma$.

The proof of the following proposition is similar to that of Proposition 3.4.

Proposition 3.10. *Let E and α be principal bundles with structure groups G and a central subgroup Γ of G respectively. Let $\theta \in \text{Aut}(G)$. An isomorphism $E \rightarrow \theta(E) \otimes \alpha$ can be identified with a (θ, Γ) -twisted automorphism of E .*

Let $A \in \text{Aut}_\theta^\Gamma(E)$. As in the previous cases, we define the function $f_A : E \rightarrow G$ by the formula

$$(3.2) \quad A(\xi) = \xi f_A(\xi) \text{ for every } \xi \in E.$$

As in Lemmas 3.5 and 3.6, we can easily prove the following.

Lemma 3.11. *Let $A \in \text{Aut}_\theta^\Gamma(E)$ and let $f_A : E \rightarrow G$ the function given by (3.2). Then*

$$f_A(\xi g) = z g^{-1} f_A(\xi) \theta(g) \text{ for } \xi \in E, g \in G,$$

where $z \in \Gamma$ depends on A, θ, ξ , and g .

Lemma 3.12. *Let $\theta_1, \theta_2 \in \text{Aut}(G)$, and $A_1 \in \text{Aut}_{\theta_1}^\Gamma(E)$ and $A_2 \in \text{Aut}_{\theta_2}^\Gamma(E)$. Then*

- (1) $A_1 A_2 \in \text{Aut}_{\theta_1 \theta_2}^\Gamma(E)$.
- (2) $f_{A_1 A_2}(\xi) = z f_{A_1}(\xi) \theta_1(f_{A_2}(\xi))$, for $\xi \in E$, where $z \in \Gamma$ depends on ξ, A_1, A_2 and θ_1 .

Proposition 3.13. *Let E be a G -bundle over a compact Riemann surface X . Let $\theta \in \text{Aut}_n(G)$ and $A \in \text{Aut}_\theta^Z(E)$ such that $A^n = f$, for a function $f : E \rightarrow Z$. Then:*

- (1) f_A defined by (3.2) maps E onto a single orbit $S(E)$ of the set S_θ^n under the action of $Z \times G$ defined in Proposition 2.22.
- (2) Every element $s \in S(E)$ defines a reduction of structure group of E to $G_{\theta'}$, where $\theta' = \text{Int}(s)\theta$ and $G_{\theta'}$ is defined as in Section 2.5.

Proof. We follow closely the proofs of Propositions 3.3 and 3.9. By Lemma 3.12,

$$f(\xi) = f_{A^n}(\xi) = z f_A(\xi) \theta(f_A(\xi)) \cdots \theta^{n-1}(f_A(\xi)),$$

where z depends on ξ . Setting $s := f_A(\xi)$, we conclude that $s \in S_\theta^n$ since $f(\xi) \in Z$.

The morphism f_A defines now a morphism $\tilde{f}_A : E \rightarrow G //_\theta (Z \times G)$, where $G //_\theta (Z \times G)$ is the GIT quotient for the action of $Z \times G$ on G given by $x \mapsto z g^{-1} x \theta(g)$ for $x, g \in G$ and $z \in Z$. Since \tilde{f}_A is constant on the fibres (Lemma 3.11), it descends to a morphism $X = E/G \rightarrow G //_\theta (Z \times G)$, which must be constant since X is projective and $G //_\theta (Z \times G)$ is affine. The rest of the argument to prove (1) is like in the proof of Proposition 3.9.

Again the proof of (2) is a straightforward generalisation of that of (2) in Proposition 3.3. Setting $S := f_A^{-1}(s)$ for $s \in S(E)$, we see that ξ and ξ' on a fibre of $E \rightarrow X$ belong to S if and only if $\xi' = \xi g$ for some $g \in G$, and $f_A(\xi) = s$ and $f_A(\xi') = s$. But, from Lemma 3.11, $s = f_A(\xi') = f_A(\xi g) = z g^{-1} f_A(\xi) \theta(g) = s$, which implies that $g \in G_{\theta'}$, with $\theta' = \text{Int}(s)\theta$. Thus the set S is acted upon transitively on fibres by $G_{\theta'}$ giving a $G_{\theta'}$ -bundle to which E is reduced. □

4. G -HIGGS BUNDLES

4.1. Moduli space of G -Higgs bundles. Let G be a complex semisimple Lie group (not necessarily connected) with Lie algebra \mathfrak{g} . Let X be a smooth projective curve over \mathbb{C} , equivalently a compact Riemann surface. A G -Higgs bundle over X is a pair (E, φ) where E is a principal G -bundle over X and φ is a section of $E(\mathfrak{g}) \otimes K$, where $E(\mathfrak{g})$ is the bundle associated to E via the adjoint representation of G , and K is the canonical bundle on X .

Two G -Higgs bundles (E, φ) and (F, ψ) are isomorphic if there is an isomorphism $f : E \rightarrow F$ such that the induced isomorphism $\text{Ad}(f) \otimes \text{Id}_K : E(\mathfrak{g}) \otimes K \rightarrow F(\mathfrak{g}) \otimes K$ sends φ to ψ .

In order to consider moduli spaces of G -Higgs bundles we need the corresponding notions of (semi,poly)stability. We briefly recall the main definitions. Our approach follows [16], where all these general notions are studied in detail.

Let \mathfrak{u} be the Lie algebra of a maximal compact subgroup U of G . Given $s \in i\mathfrak{u}$,

$$(4.1) \quad P_s = \{g \in G : e^{ts}ge^{-ts} \text{ is bounded as } t \rightarrow \infty\},$$

is a parabolic subgroup of G , whose corresponding parabolic subalgebra of \mathfrak{g} is

$$\mathfrak{p}_s = \{v \in \mathfrak{g} : \text{Ad}(e^{ts})(v) \text{ is bounded as } t \rightarrow \infty\}.$$

If, moreover, we define

$$(4.2) \quad L_s = \{g \in G : \lim_{t \rightarrow \infty} e^{ts}ge^{-ts} = g\}$$

then $L_s \subset P_s$ is a Levi subgroup of P_s , and

$$\mathfrak{l}_s = \{v \in \mathfrak{g} : \lim_{t \rightarrow \infty} \text{Ad}(e^{ts})(v) = 0\}$$

is the corresponding Levi subalgebra of \mathfrak{p}_s .

When G is connected, every parabolic subgroup P is of the form (4.1) for some $s \in i\mathfrak{u}$; the same holds for the Levi subgroups. For G non-connected that may not be the case (cf. [31, Remark 5.3]). However, in order to define semistability, the parabolic subgroups which need to be considered are precisely the ones of the form (4.1). Hence, for simplicity, and when no explicit mention to $s \in i\mathfrak{u}$ is needed, we refer to these as the parabolic subgroups of G , keeping in mind that we mean the groups defined by (4.1). We will do the same for the Levi subgroups, referring to (4.2).

Let P be a parabolic subgroup of G . A character of the Lie algebra \mathfrak{p} of P is a complex linear map $\mathfrak{p} \rightarrow \mathbb{C}$ which factors through $\mathfrak{p}/[\mathfrak{p}, \mathfrak{p}]$. Let $\mathfrak{l} \subset \mathfrak{p}$ be the corresponding Levi subalgebra and let $\mathfrak{z}_{\mathfrak{l}}$ be the centre of \mathfrak{l} . Then, one has that $(\mathfrak{p}/[\mathfrak{p}, \mathfrak{p}])^* \cong \mathfrak{z}_{\mathfrak{l}}^*$, so the characters of \mathfrak{p} are indeed classified by elements of $\mathfrak{z}_{\mathfrak{l}}^*$. Using the Killing form a character $\chi \in \mathfrak{z}_{\mathfrak{l}}^*$ of \mathfrak{p} is uniquely determined by an element $s_{\chi} \in \mathfrak{z}_{\mathfrak{l}}$. Indeed, it can be shown that $\mathfrak{z}_{\mathfrak{l}} \subset i\mathfrak{u}$, so that $s_{\chi} \in i\mathfrak{u}$. Now, the character χ of \mathfrak{p} is said to be **antidominant** if $\mathfrak{p} \subset \mathfrak{p}_{s_{\chi}}$ and **strictly antidominant** if $\mathfrak{p} = \mathfrak{p}_{s_{\chi}}$. Given a character $\chi : P \rightarrow \mathbb{C}^*$ of P , denote by χ_* the corresponding character of \mathfrak{p} . We say that χ is **(strictly) antidominant** if χ_* is. An element $s \in i\mathfrak{u}$ defines clearly a character χ_s of \mathfrak{p}_s since $\langle s, [\mathfrak{p}_s, \mathfrak{p}_s] \rangle = 0$ (where $\langle \cdot, \cdot \rangle$ is the Killing form) which is of course strictly antidominant.

Let $\chi : P \rightarrow \mathbb{C}^*$ be an antidominant character of P and Let $\sigma \in \Gamma(X, E/P)$ be a reduction of the structure group of E to P . Denote by $E_\sigma \subset E$ the corresponding holomorphic P -bundle. The **degree** of E with respect to σ and χ , denoted by $\deg(E)(\sigma, \chi)$, is the degree of the line bundle obtained by extending the structure group of E_σ through χ . In other words,

$$(4.3) \quad \deg(E)(\sigma, \chi) = \deg(E_\sigma \times_\chi \mathbb{C}^*).$$

Given $s \in i\mathfrak{u}$ and a reduction of structure group of E to P_s we can also define $\deg(E)(\sigma, s)$, even though χ_s may not lift to a character of P_s . One way to do this is to use Chern–Weil theory. This definition is more natural when considering gauge-theoretic equations as we do below. For this, define $U_s = U \cap L_s$ and $\mathfrak{u}_s = \mathfrak{u} \cap \mathfrak{l}_s$. Then U_s is a maximal compact subgroup of L_s , so the inclusions $U_s \subset L_s$ is a homotopy equivalence. Since the inclusion $L_s \subset P_s$ is also a homotopy equivalence, given a reduction σ of the structure group of E to P_s one can further restrict the structure group of E to U_s in a unique way up to homotopy. Denote by E'_σ the resulting U_s principal bundle. Consider now a connection A on E'_σ and let $F_A \in \Omega^2(X, E'_\sigma(\mathfrak{u}_s))$ be its curvature. Then $\chi_s(F_A)$ is a 2-form on X with values in $i\mathbb{R}$, and

$$(4.4) \quad \deg(E)(\sigma, s) := \frac{i}{2\pi} \int_X \chi_s(F_A).$$

This coincides with $\deg(E)(\sigma, \tilde{\chi}_s)$ as defined in (4.3) when χ_s can be lifted to a character $\tilde{\chi}_s$ of P_s .

Definition 4.1. A G -Higgs bundle (E, φ) over X is:

semistable if $\deg(E)(\sigma, \chi) \geq 0$, for any parabolic subgroup P of G , any non-trivial antidominant character χ of P and any reduction of structure group σ of E to P such that $\varphi \in H^0(X, E_\sigma(\mathfrak{p}) \otimes K)$.

stable if $\deg(E)(\sigma, \chi) > 0$, for any non-trivial parabolic subgroup P of G , any non-trivial antidominant character χ of P and any reduction of structure group σ of E to P such that $\varphi \in H^0(X, E_\sigma(\mathfrak{p}) \otimes K)$.

polystable if it is semistable and if $\deg(E)(\sigma, \chi) = 0$, for some parabolic subgroup $P \subset G$, some non-trivial strictly antidominant character χ of P and some reduction of structure group σ of E to P such that $\varphi \in H^0(X, E_\sigma(\mathfrak{p}) \otimes K)$, then there is a further holomorphic reduction of structure group σ_L of E_σ to the Levi subgroup L of P such that $\varphi \in H^0(X, E_{\sigma_L}(\mathfrak{l}) \otimes K)$.

Remark 4.2. (1) A G -Higgs bundle with $\varphi = 0$ is a holomorphic principal G -bundle and a (semi)stability condition for these objects over compact Riemann surfaces was established by Ramanathan in [39]. One has a direct generalisation of Ramanathan's condition to the G -Higgs bundle case for G complex (see for example [5, 16]).

(2) The notion of G -Higgs bundle given above makes sense also when G is more generally a complex reductive Lie group (in fact any complex Lie group). In the general reductive case, however, the stability criteria has to be modified by replacing $\deg(E)(\sigma, \chi)$ in Definition 4.1 with $\deg(E)(\sigma, \chi) - \langle \alpha, s_{\chi_*} \rangle$, where $\langle \cdot, \cdot \rangle$ is an invariant \mathbb{C} -bilinear pairing on \mathfrak{g} extending the Killing form on the semisimple part, and α is an element in $\mathfrak{i}\mathfrak{z}_{\mathfrak{u}}$ determined by the topology of the G -bundle E . There is no discrepancy with [39, 5] where the reductive case is treated and there is no parameter α . The reason is that these authors consider

characters which are trivial on the centre of G , and hence the corresponding characters on the Lie algebra are orthogonal to α with respect to the pairing $\langle \cdot, \cdot \rangle$.

Let $\mathcal{M}(G)$ be **moduli space of semistable G -Higgs bundles**. As usual, the moduli space $\mathcal{M}(G)$ can also be viewed as parametrizing isomorphism classes of polystable G -Higgs bundles. The space $\mathcal{M}(G)$ has the structure of a quasi-projective variety, as one can see from the Schmitt's general Geometric Invariant Theory construction (cf. [41]). For related constructions see [35, 46, 47]. If we fix the topological class c of E we can consider $\mathcal{M}_c(G) \subset \mathcal{M}(G)$, the moduli space of semistable G -Higgs bundles with fixed topological class c . If G is connected the topological class is given by an element of $c \in \pi_1(G)$. In this situation it is well-known ([30, 12]) that $\mathcal{M}_c(G)$ is non-empty and connected. A Morse-theoretic proof of this fact has been given recently in [18], where the connectedness and non-emptiness of $\mathcal{M}_c(G)$ is also proved when G is a non-connected complex reductive Lie group.

Let (E, φ) be a Higgs bundle. Since the induced action of $Z(G)$ on $E(\mathfrak{g})$ is trivial, they preserve φ and we see that $Z(G)$ is a central subgroup of $\text{Aut}(E, \varphi)$, and that the quotient is finite if (E, φ) is stable.

Definition 4.3. *A G -Higgs bundle (E, φ) is said to be **simple** if $\text{Aut}(E, \varphi)$ is $Z(G)$.*

Remark 4.4. When $G = \text{GL}(n, \mathbb{C})$ or $SL(n, \mathbb{C})$, every stable bundle is simple. This is not true in general. In the case $G = \text{SO}(n, \mathbb{C})$, the direct sum of two non-isomorphic orthogonal bundles is still stable but is in general not simple. The phenomenon of stable bundles not being simple has to do with the coefficients of the highest root of a simple Lie algebra being not 1. This phenomenon is related to the existence of elements of finite order in the group G whose centralizers are semisimple (not just reductive). One can show that $\text{GL}(n, \mathbb{C})$ or $SL(n, \mathbb{C})$ are the only cases for which this does not happen.

We have the following (see [16]).

Proposition 4.5. *A G -Higgs bundle which is stable and simple defines a smooth point in $\mathcal{M}(G)$.*

4.2. G -Higgs bundles and Hitchin equations. As above, let G be a complex semisimple Lie group. Let $U \subset G$ be a maximal compact subgroup defined by a conjugation τ of G . Let (E, φ) be a G -Higgs bundle over a compact Riemann surface X . By a slight abuse of notation, we shall denote the C^∞ -objects underlying E and φ by the same symbols. In particular, the Higgs field can be viewed as a $(1, 0)$ -form $\varphi \in \Omega^{1,0}(E(\mathfrak{g}))$ with values in $E(\mathfrak{g})$. Given a C^∞ reduction of structure group h of the principal G -bundle E to U , we can define

$$(4.5) \quad \tau_h : \Omega^{1,0}(E(\mathfrak{g})) \longrightarrow \Omega^{0,1}(E(\mathfrak{g}))$$

the isomorphism induced by the compact conjugation of \mathfrak{g} defined by τ , combined with the complex conjugation on complex 1-forms.

we denote by F_h the curvature of the unique connection compatible with h and the holomorphic structure on E (see [3]). One has the following.

Theorem 4.6. *Let (E, φ) be a G -Higgs bundle. There is a reduction h of structure group of E from G to U that satisfies the Hitchin equation*

$$F_h - [\varphi, \tau_h(\varphi)] = 0$$

if and only if (E, φ) is polystable.

Theorem 4.6 was proved by Hitchin [24] for $G = \mathrm{SL}(2, \mathbb{C})$, and by Simpson in [44, 45] for the general case (see also [16]).

Remark 4.7. Theorem 4.6 can be extended to the case in which G is a complex reductive Lie group. In this situation the Hitchin equation is replaced with $F_h - [\varphi, \tau_h(\varphi)] = -i\alpha\omega$, where $\alpha \in i\mathfrak{z}_u$ is determined by the topology of E , via Chern–Weil theory from the equation and ω is a volume form of X .

From the point of view of moduli spaces it is convenient to fix a C^∞ principal U -bundle \mathbf{E}_U and study the moduli space of solutions to **Hitchin's equations** for a pair (A, φ) consisting of an U -connection A and a section $\varphi \in \Omega^{1,0}(X, \mathbf{E}_U(\mathfrak{g}))$:

$$(4.6) \quad \begin{aligned} F_A - [\varphi, \tau_h(\varphi)] &= 0 \\ \bar{\partial}_A \varphi &= 0. \end{aligned}$$

Here d_A is the covariant derivative associated to A , and $\bar{\partial}_A$ is the $(0, 1)$ part of d_A . The $(0, 1)$ part of d_A defines a holomorphic structure on \mathbf{E}_U . The gauge group \mathcal{U} of \mathbf{E}_U , i.e. the group of automorphisms of \mathbf{E}_U acts on the space of solutions and the moduli space of solutions is

$$\mathcal{M}_c^{\mathrm{gauge}}(G) := \{(A, \varphi) \text{ satisfying (4.6)}\} / \mathcal{U},$$

where c is the topological type of the bundle \mathbf{E}_U . The irreducible solutions define smooth points in $\mathcal{M}_c^{\mathrm{gauge}}(G)$. Now, if $\mathcal{M}_c(G)$ is the moduli space of G -Higgs bundles of topological type c , i.e. the G -Higgs bundles whose underlying C^∞ bundle is the G -bundle obtained from \mathbf{E}_U by extension of the structure group to G , Theorem 4.6 can be reformulated as follows.

Theorem 4.8. *There is a homeomorphism*

$$\mathcal{M}_c(G) \cong \mathcal{M}_c^{\mathrm{gauge}}(G).$$

Moreover, the irreducible solutions in $\mathcal{M}_c^{\mathrm{gauge}}(G)$ correspond with the stable and simple Higgs bundles in $\mathcal{M}_c(G)$.

To explain this correspondence we interpret the moduli space of G -Higgs bundles in terms of pairs $(\bar{\partial}_E, \varphi)$ consisting of a $\bar{\partial}$ -operator (holomorphic structure) $\bar{\partial}_E$ on the C^∞ principal G -bundle \mathbf{E}_G obtained from \mathbf{E}_U by the extension of structure group $U \hookrightarrow G$, and $\varphi \in \Omega^{1,0}(X, \mathbf{E}_G(\mathfrak{g}))$ satisfying $\bar{\partial}_E \varphi = 0$. Such pairs are in one-to-one correspondence with G -Higgs bundles (E, φ) , where E is the holomorphic G -bundle defined by the operator $\bar{\partial}_E$ on \mathbf{E}_G . The equation $\bar{\partial}_E \varphi = 0$ is equivalent to the condition that $\varphi \in H^0(X, E(\mathfrak{g}) \otimes K)$. The moduli space of polystable G -Higgs bundles of topological type given by \mathbf{E}_U can now be identified with the orbit space

$$\{(\bar{\partial}_E, \varphi) : \bar{\partial}_E \varphi = 0 \text{ which are polystable}\} / \mathcal{G},$$

where \mathcal{G} is the gauge group of \mathbf{E}_G , which is in fact the complexification of \mathcal{U} . Since there is a one-to-one correspondence between U -connections on \mathbf{E}_U and $\bar{\partial}$ -operators on \mathbf{E}_G , the correspondence given in Theorem 4.8 can be reformulated by saying that in the \mathcal{G} -orbit of a polystable G -Higgs bundle $(\bar{\partial}_{E_0}, \varphi_0)$ we can find another Higgs bundle $(\bar{\partial}_E, \varphi)$ whose corresponding pair (d_A, φ) satisfies the Hitchin equation $F_A - [\varphi, \tau_h(\varphi)] = 0$ with this pair (d_A, φ) being unique up to U -gauge transformations.

4.3. Hyperkähler structure. We will see now that the smooth locus of the moduli space $\mathcal{M}(G)$ has a hyperkähler structure. We follow [24]. For this, recall first that a hyperkähler manifold is a differentiable manifold M equipped with a Riemannian metric g and complex structures J_i , $i = 1, 2, 3$ satisfying the quaternion relations $J_i^2 = -I$, $J_3 = J_1 J_2 = -J_2 J_1$, $J_2 = -J_1 J_3 = J_3 J_1$ and $J_1 = J_2 J_3 = -J_3 J_2$ such that if we define $\omega_i(\cdot, \cdot) = g(J_i \cdot, \cdot)$, then (g, J_i, ω_i) is a Kähler structure on M . Let Ω_i denote the holomorphic symplectic structure on $\mathcal{M}(G)$ with respect to the complex structure J_i . That is, $\Omega_1 = \omega_2 + \sqrt{-1}\omega_3$, $\Omega_2 = \omega_3 + \sqrt{-1}\omega_1$ and $\Omega_3 = \omega_1 + \sqrt{-1}\omega_2$.

We explain now how the smooth locus of the moduli space $\mathcal{M}_c^{\text{gauge}}(G)$, and hence that of $\mathcal{M}_c(G)$, can be obtained as hyperkähler quotient. For this, let \mathbf{E}_G be a smooth principal G -bundle over X , with topological class c , and let \mathbf{E}_U be a fixed reduction of \mathbf{E}_G to the maximal compact subgroup U . The set \mathcal{A} of U -connections on \mathbf{E}_U is an affine space modelled on $\Omega^1(X, \mathbf{E}(\mathfrak{u}))$. Via the Chern correspondence \mathcal{A} is in one-to-one correspondence with the set \mathcal{C} of holomorphic structures on \mathbf{E}_G [3], which is an affine space modelled on $\Omega^{0,1}(X, \mathbf{E}_G(\mathfrak{g}))$. Let us denote $\Omega = \Omega^{1,0}(X, \mathbf{E}_G(\mathfrak{g}))$. We consider $\mathcal{X} = \mathcal{A} \times \Omega$. Via the identification $\mathcal{A} \cong \mathcal{C}$, we have for $\alpha \in \Omega^{0,1}(X, \mathbf{E}_G(\mathfrak{g}))$ and $\psi \in \Omega^{1,0}(X, \mathbf{E}_G(\mathfrak{g}))$ the following three complex structures on \mathcal{X} :

$$\begin{aligned} J_1(\alpha, \psi) &= (\sqrt{-1}\alpha, \sqrt{-1}\psi) \\ J_2(\alpha, \psi) &= (-\sqrt{-1}\tau_h(\psi), \sqrt{-1}\tau_h(\alpha)) \\ J_3(\alpha, \psi) &= (\tau_h(\psi), -\tau_h(\alpha)), \end{aligned}$$

where τ_h is the conjugation on \mathfrak{g} defining its compact form \mathfrak{u} (determined fibrewise by a reduction h to \mathbf{E}_U), combined with the complex conjugation on complex 1-forms.

One has also a Riemannian metric g defined on \mathcal{X} : for $\alpha \in \Omega^{0,1}(X, \mathbf{E}_G(\mathfrak{g}))$ and $\psi \in \Omega^{1,0}(X, \mathbf{E}_G(\mathfrak{g}))$,

$$g((\alpha, \psi), (\alpha, \psi)) = -2\sqrt{-1} \int_X B(\tau_h(\alpha), \alpha) + B(\psi, \tau_h(\psi)),$$

where B is the Killing form.

Clearly, J_i , $i = 1, 2, 3$, satisfy the quaternion relations, and define a hyperkähler structure on \mathcal{X} , with Kähler forms $\omega_i(\cdot, \cdot) = g(J_i \cdot, \cdot)$, $i = 1, 2, 3$. As shown in [24], the action of the gauge group \mathcal{U} on \mathcal{X} preserves the hyperkähler structure and there are moment maps given by

$$\mu_1(A, \varphi) = F_A - [\varphi, \tau_h(\varphi)], \quad \mu_2(A, \varphi) = \text{Re}(\bar{\partial}_E \varphi), \quad \mu_3(A, \varphi) = \text{Im}(\bar{\partial}_E \varphi).$$

We have that $\mu^{-1}(0)/\mathcal{U}$, where $\mu = (\mu_1, \mu_2, \mu_3)$ is the moduli space of solutions to the Hitchin's equations (4.6). In particular, if we consider the irreducible solutions (equivalently, smooth) $\mu_*^{-1}(0)$ we have that

$$\mu_*^{-1}(0)/\mathcal{U}$$

is a hyperkähler manifold which, by Theorem 4.8, is homeomorphic to the subvariety of smooth points $\mathcal{M}(G)_{ss} \subset \mathcal{M}(G)$ consisting of stable and simple G -Higgs bundles with the topological class of \mathbf{E}_G .

4.4. Automorphisms of $\mathcal{M}(G)$. Let G be a complex semisimple Lie group and let $\text{Aut}(G)$, $\text{Int}(G)$ and $\text{Out}(G)$ be as defined in Section 2.1.

Let $\theta \in \text{Aut}(G)$, and let (E, φ) be a G -Higgs bundle over X . We define the G -Higgs bundle $(\theta(E), \theta(\varphi))$ by taking:

$$\theta(E) := E \times_{\theta} G,$$

and using that $\theta(E)(\mathfrak{g}) \cong E(\mathfrak{g})$ and the differential $d\theta \in \text{Aut}(\mathfrak{g})$, associating to the section φ of $E(\mathfrak{g}) \otimes K$, a section of $\theta(E)(\mathfrak{g}) \otimes K$, that we will denote by $\theta(\varphi)$. It is clear that if $\theta \in \text{Int}(G)$, the Higgs bundle $(\theta(E), \theta(\varphi))$ is isomorphic to (E, φ) . Hence the group $\text{Out}(G)$ acts on the set of isomorphism classes of G -Higgs bundles. It is easy to show that stability, semistability and polystability are preserved by the action of $\text{Aut}(G)$ and hence $\text{Out}(G)$ acts on the moduli space of G -Higgs bundles $\mathcal{M}(G)$.

We also have an action of \mathbb{C}^* on the set of G -Higgs bundles defined as follows. Let (E, φ) be a G -Higgs bundle and $\lambda \in \mathbb{C}^*$, we define

$$\lambda(E, \varphi) := (E, \lambda\varphi).$$

It is immediate that stability, semistability and polystability are preserved by this action and this defines an action of \mathbb{C}^* on $\mathcal{M}(G)$.

Let $Z := Z(G)$ be the centre of G . Since Z is abelian, $H^1(X, Z)$ is a group and it is identified with the set of isomorphism classes of principal Z -bundles over X . For $\alpha, \beta \in H^1(X, Z)$ we consider first the $(Z \times Z)$ -bundle given by the fibre product $\alpha \times_X \beta$ over X . Now, via the group operation $Z \times Z \xrightarrow{m} Z$ we can associate to it the Z -bundle

$$\alpha \otimes \beta := (\alpha \times_X \beta) \times_m Z,$$

defining the group structure of $H^1(X, Z)$.

Let E be a principal G -bundle and let $\alpha \in H^1(X, Z)$ be a principal Z -bundle. The fibre product $E \times_X \alpha$ has the structure of a principal $(G \times Z)$ -bundle. Combining this with the action of Z on G given by multiplication $G \times Z \xrightarrow{m} G$, by extension of structure group we associate to E and α the principal G -bundle $E \otimes \alpha := (E \times_X \alpha) \times_m G$.

Since G is semisimple Z is finite, and the topological type of E and $E \otimes \alpha$ is the same.

It is clear that $E(\mathfrak{g}) = (E \otimes \alpha)(\mathfrak{g})$ and hence we can associate to a G -Higgs bundle (E, φ) and $\alpha \in H^1(X, Z)$ the G -Higgs bundle defined by

$$\alpha \cdot (E, \varphi) := (E \otimes \alpha, \varphi).$$

Again it is immediate to show that this defines an action of $H^1(X, Z)$ on the moduli space of G -Higgs bundles, $\mathcal{M}(G)$.

Every automorphism of G restricts to an automorphism of Z . Inner automorphisms of G induce the identity automorphism on Z , defining an action of $\text{Out}(G)$ on Z , $\text{Out}(G) \times Z \rightarrow Z$, $(\sigma, \lambda) \mapsto \sigma(\lambda)$. This induces an action of $\text{Out}(G)$ on $H^1(X, Z)$. We consider the semidirect product $H^1(X, Z) \rtimes \text{Out}(G)$ defined by

$$(\beta, b) \cdot (\alpha, a) = (\beta \otimes b(\alpha), ba).$$

We thus notice that the group $H^1(X, Z) \rtimes \text{Out}(G)$ acts on $\mathcal{M}(G)$ in the following way: if (E, φ) is a polystable G -Higgs bundle and $(\alpha, a) \in H^1(X, Z) \rtimes \text{Out}(G)$, then

$$(\alpha, a) \cdot (E, \varphi) = (a(E) \otimes \alpha, a(\varphi)).$$

We also have the semidirect product

$$H^1(X, Z) \rtimes \text{Aut}(X)$$

defined by

$$(\alpha_2, f_2)(\alpha_1, f_1) = (\alpha_2 \otimes f_2^* \alpha_1, f_2 f_1).$$

One can show that we have an action of $H^1(X, Z) \rtimes \text{Aut}(X)$ on $\mathcal{M}(G)$ given by

$$(\alpha, f)(E, \varphi) = (f^* E \otimes \alpha, f^* \varphi),$$

where $(E, \varphi) \in \mathcal{M}(G)$ and $(\alpha, f) \in H^1(X, Z) \rtimes \text{Aut}(X)$.

Combining the preceding actions we obtain an action of the group

$$H^1(X, Z) \rtimes (\text{Out}(G) \times \text{Aut}(X)) \times \mathbb{C}^*$$

on $\mathcal{M}(G)$.

Remark 4.9. The above discussion can be extended to the case where G is reductive. We just need to replace $H^1(X, Z)$ with $H^1(X, \underline{Z})$, where \underline{Z} be the sheaf of local Z -functions on X . Of course now the topological type of E and $E \otimes \alpha$ is the same only if $\alpha \in H^1(X, \underline{Z})$ has trivial topological class.

5. G -HIGGS BUNDLES AND INVOLUTIONS OF G

In this section we will assume that G is a complex semisimple Lie group. To study the involutions of $\mathcal{M}(G)$ considered in Section 7, we introduce now a class of Higgs bundles defined by an involution $\theta \in \text{Aut}_2(G)$. These involve the subgroups G^θ and G_θ of G defined in Section 2.4.

5.1. Higgs bundles defined by involutions of G . A (G^θ, \pm) -Higgs bundle over X is a pair (E, φ) where E is a principal G^θ -bundle over X and φ is a section of $E(\mathfrak{g}^\pm) \otimes K$, where $E(\mathfrak{g}^\pm)$ is the bundle associated to E via the representation $\iota^\pm : G_\theta \rightarrow \text{GL}(\mathfrak{g}^\pm)$ defined in Proposition 2.21, and, as above, K is the canonical bundle on X . It is clear that a $(G^\theta, +)$ -Higgs bundle is simply a G^θ -Higgs bundle as defined in Section 4.1. Let τ be a fixed compact conjugation of G defining a maximal compact subgroup $U := G^\tau$, such that $\tau\theta = \theta\tau$. Let $\sigma := \theta\tau$. Notice that, as explained in Section 2.6, G^θ is semisimple unless G^σ has factors of Hermitian type, in which case it is generally a reductive group. The notions of (semi)stability and polystability for $(G^\theta, +)$ -Higgs bundles are hence given in Definition 4.1 (see (2) in Remark 4.2 for the case in which G^θ is reductive). The case of

$(G^\theta, -)$ -Higgs bundles requires a new definition. For convenience, however, we will give a definition that accommodates both cases simultaneously.

The group U^σ defined in Section 2.4, is a maximal compact subgroup of G^θ . Its Lie algebra is \mathfrak{u}^σ , where \mathfrak{u} is the Lie algebra of U , and we are denoting by σ also the antilinear involution induced in \mathfrak{u} by σ . Clearly, $(\mathfrak{u}^\sigma)^\mathbb{C} = \mathfrak{g}^\theta = \mathfrak{g}^+$.

Given $s \in i\mathfrak{u}^\sigma$, as explained in Section 4.1,

$$(5.1) \quad P_s = \{g \in G^\theta : e^{ts}ge^{-ts} \text{ is bounded as } t \rightarrow \infty\},$$

is a parabolic subgroup of G^θ .

We also define

$$(5.2) \quad \mathfrak{g}_s^\pm = \{v \in \mathfrak{g}^\pm : \text{Ad}(e^{ts})(v) \text{ is bounded as } t \rightarrow \infty\}.$$

$$(5.3) \quad \mathfrak{g}_{s,0}^\pm = \{v \in \mathfrak{g}^\pm : \lim_{t \rightarrow \infty} \text{Ad}(e^{ts})(v) = 0\}$$

Clearly $\mathfrak{g}_s^+ = \mathfrak{p}_s$, the Lie algebra of P_s , and $\mathfrak{g}_{s,0}^+$ is the Levi part \mathfrak{l}_s of \mathfrak{p}_s .

As in Section 4.1, the only parabolic subgroups of G^θ that we will consider are subgroups of form P_s for some $s \in i\mathfrak{u}^\sigma$.

We consider the non-degenerate invariant \mathbb{C} -bilinear pairing $\langle \cdot, \cdot \rangle$ on \mathfrak{g}^θ induced by the Killing form on \mathfrak{g} .

Let $\mathfrak{z}(\mathfrak{u}^\sigma)$ be the centre of \mathfrak{u}^σ . As explained in Section 2.6, $\mathfrak{z}(\mathfrak{u}^\sigma) = 0$ unless there is a Hermitian factor in \mathfrak{g}^σ . In this situation the natural stability criteria depends on a parameter $\alpha \in i\mathfrak{z}(\mathfrak{u}^\sigma)$ (see [16] for details).

We define the subalgebra \mathfrak{u}_\pm^σ as follows. Consider the decomposition $\mathfrak{u}^\sigma = \mathfrak{z}(\mathfrak{u}^\sigma) + [\mathfrak{u}^\sigma, \mathfrak{u}^\sigma]$, and the representation $d\iota^\pm = \text{ad} : \mathfrak{u}^\sigma \rightarrow \text{End}(\mathfrak{g}^\pm)$. Let $\mathfrak{z}'_\pm = \ker(d\iota^\pm|_{\mathfrak{z}(\mathfrak{u}^\sigma)})$ and take \mathfrak{z}''_\pm such that $\mathfrak{z}(\mathfrak{u}^\sigma) = \mathfrak{z}'_\pm + \mathfrak{z}''_\pm$. Define the subalgebra $\mathfrak{u}_\pm^\sigma := \mathfrak{z}''_\pm + [\mathfrak{u}^\sigma, \mathfrak{u}^\sigma]$. The subindex \pm denotes that we have taken away the part of the centre $\mathfrak{z}(\mathfrak{u}^\sigma)$ acting trivially via the isotropy representation $d\iota^\pm$. Note that since $d\iota^+$ is the adjoint representation of \mathfrak{u}^σ , $\mathfrak{z}' = \mathfrak{z}(\mathfrak{u}^\sigma)$ and $\mathfrak{u}_+^\sigma = [\mathfrak{u}^\sigma, \mathfrak{u}^\sigma]$.

We have now all the ingredients to define stability.

Definition 5.1. Let $\alpha \in i\mathfrak{z}(\mathfrak{u}^\sigma)$. A (G^θ, \pm) -Higgs bundle (E, φ) over X is:

α -semistable if for any $s \in i\mathfrak{u}^\sigma$ and any reduction of structure group σ of E to P_s such that $\varphi \in H^0(X, E_\sigma(\mathfrak{g}_s^\pm) \otimes K)$ we have that $\deg(E)(\sigma, s) - \langle \alpha, s \rangle \geq 0$.

α -stable if for any $s \in i\mathfrak{u}_\pm^\sigma$ and any reduction of structure group σ of E to P_s such that $\varphi \in H^0(X, E_\sigma(\mathfrak{g}_s^\pm) \otimes K)$ we have that $\deg(E)(\sigma, s) - \langle \alpha, s \rangle > 0$.

α -polystable if it is α -semistable and $s \in i\mathfrak{u}_\pm^\sigma$ and any reduction of structure group σ of E to P_s such that $\varphi \in H^0(X, E_\sigma(\mathfrak{g}_s^\pm) \otimes K)$ and $\deg(E)(\sigma, s) - \langle \alpha, s \rangle = 0$ there is a further reduction of structure group σ_L of E_σ to the Levi subgroup L_s of P_s such that $\varphi \in H^0(X, E_{\sigma_L}(\mathfrak{g}_{s,0}^\pm) \otimes K)$.

Let $\mathcal{M}^\alpha(G^\theta, \pm)$ be the **moduli space of polystable (G^θ, \pm) -Higgs bundles**. The construction of the moduli space of pairs given by Schmitt using Geometric Invariant Theory (cf. [41]) applies also to these moduli spaces.

Remark 5.2. When $\alpha = 0$ we will simply say stability to refer to 0-stability and will drop the index α denoting the moduli space by $\mathcal{M}(G^\theta, \pm)$.

Remark 5.3. As mentioned above, a $(G^\theta, +)$ -Higgs bundle (E, φ) is simply a G^θ -Higgs bundle, and, as pointed out in Remark 4.2, the parameter α is determined by the topology of E . This is definitely not the case for $(G^\theta, -)$ -Higgs bundles, where α can take different values. This happens precisely when the real form defined by σ has a factor of Hermitian type. In fact the possible values of the parameter α are governed by a Milnor–Wood type inequality (see [4]).

As for G -Higgs bundles, there are relevant Hitchin equations linked to a (G^θ, \pm) -Higgs bundle. With the same notation as in Theorem 4.6, One has the following.

Theorem 5.4. *Let (E, φ) be a (G^θ, \pm) -Higgs bundle. There is a reduction h of structure group of E from G^θ to U^σ that satisfies the Hitchin equation*

$$F_h - [\varphi, \tau_h(\varphi)] = -i\alpha\omega$$

if and only if (E, φ) is α -polystable.

The proof of Theorem 5.5 is given in [16] (the $(G^\theta, +)$ -case is essentially given by Theorem 4.6).

For reasons that will be apparent in Section 6, $(G^\theta, -)$ -Higgs bundles are referred in the literature as G^σ -Higgs bundles.

Recall from Section 2.4 the short exact sequence (2.9):

$$(5.4) \quad 1 \longrightarrow G^\theta \longrightarrow G_\theta \longrightarrow \Gamma_\theta \longrightarrow 1.$$

Similarly to (G^θ, \pm) -Higgs bundles we can consider (G_θ, \pm) -Higgs bundles. A (G_θ, \pm) -**Higgs bundle** over X is a pair (E, φ) where E is a principal G_θ -bundle over X and φ is a section of $E(\mathfrak{g}^\pm) \otimes K$, where $E(\mathfrak{g}^\pm)$ is the bundle associated to E via the representation $\iota_\pm : G_\theta \rightarrow \mathrm{GL}(\mathfrak{g}^\pm)$ defined in Proposition 2.21. Stability can be defined as in Definition 5.1, where now the parabolic subgroups defined by elements $s \in i\mathfrak{u}^\sigma$ are parabolic subgroups of G_θ . Note that the maximal compact subgroup of G_θ is the group U_σ defined in Section 2.4, and its Lie algebra is also \mathfrak{u}^σ . We denote the moduli space of α -polystable such objects as $\mathcal{M}^\alpha(G_\theta, \pm)$ — or simply $\mathcal{M}(G_\theta, \pm)$ if $\alpha = 0$.

A correspondence theorem analogous to Theorem 5.4 also exists for these objects ([16]):

Theorem 5.5. *Let (E, φ) be a (G_θ, \pm) -Higgs bundle. There is a reduction h of structure group of E from G_θ to U_σ that satisfies the Hitchin equation*

$$F_h - [\varphi, \tau_h(\varphi)] = -i\alpha\omega$$

if and only if (E, φ) is α -polystable.

Consider the map

$$(5.5) \quad \gamma_\theta : H^1(X, \underline{G}_\theta) \longrightarrow H^1(X, \Gamma_\theta)$$

induced by the homomorphism $G_\theta \longrightarrow \Gamma_\theta$. This map associates to every G_θ -bundle an invariant in $H^1(X, \Gamma_\theta)$. If we fix $\gamma \in H^1(X, \Gamma_\theta)$ we can consider the moduli space $\mathcal{M}_\gamma^\alpha(G_\theta, \pm)$ of α -polystable elements (E, φ) as above such that $\gamma_\theta(E) = \gamma$ (or simply $\mathcal{M}_\gamma(G_\theta, \pm)$ for the moduli space 0-polystable elements). Higgs bundles (E, φ) such that $\gamma(E) = e \in H^1(X, \Gamma_\theta)$ — the identity element — have the property that the structure group of E reduces to G^θ . We have the following.

Proposition 5.6. *The group Γ_θ acts on $\mathcal{M}^\alpha(G^\theta, \pm)$, and*

$$\mathcal{M}_e^\alpha(G_\theta, \pm) = \mathcal{M}^\alpha(G^\theta, \pm)/\Gamma_\theta.$$

Proof. We have the characteristic homomorphism $\Gamma_\theta \rightarrow \text{Out}(G^\theta) = \text{Aut}(G^\theta)/\text{Int}(G^\theta)$, which then defines an action on $H^1(X, \underline{G}_\theta)$. So if E is G^θ -bundle and $\gamma \in \Gamma_\theta$, we have another G^θ -bundle $\gamma(E)$. To obtain a section of $\gamma(E) \otimes \mathfrak{g}^\pm \otimes K$ out of the Higgs field $\varphi \in E(\mathfrak{g}^\pm) \otimes K$ we observe that Γ_θ acts also on \mathfrak{g}^\pm in a way that $\iota^\pm : G^\theta \rightarrow \text{GL}(\mathfrak{g}^\pm)$ is Γ_θ -equivariant. That Γ_θ acts on \mathfrak{g}^+ is clear since $\mathfrak{g}^+ = \mathfrak{g}^\theta$. The action on \mathfrak{g}^- comes from the fact that $\Gamma_\theta = \Gamma_\sigma$ as stated in Proposition 2.17, and we have the characteristic homomorphism $\Gamma_\sigma \rightarrow \text{Out}(G^\sigma) = \text{Aut}(G^\sigma)/\text{Int}(G^\sigma)$, which combined with the previous one gives the compatibility of the action of Γ_θ with the Cartan decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$. The preservation of semistability, stability and polystability under this action is clear.

Let now $(E, \varphi) \in \mathcal{M}^\alpha(G^\theta, \pm)$. We can define a (G_θ, \pm) -Higgs bundle by extending the structure group of E to G_θ obtaining a G_θ -bundle E_{G_θ} . The extended Higgs field φ_{G_θ} is in fact φ itself since $E(\mathfrak{g}^\pm) = E_{G_\theta}(\mathfrak{g}^\pm)$. This map preserves polystability as can be seen easily from Theorem 5.4. Notice that a reduction of the structure group of a G^θ -bundle E to the maximal compact subgroup U^σ is a smooth section of the bundle $E(G^\theta/U^\sigma)$ and since these symmetric spaces G^θ/U^σ and G_θ/U_σ coincide $E(G^\theta/U^\sigma) = E_{G_\theta}(G_\theta/U_\sigma)$. Since the different reductions of a G_θ -bundle whose with trivial class in $H^1(X, \Gamma_\theta)$ are given by $H^0(X, \Gamma_\theta) \cong \Gamma_\theta$ the map defined above descends to give the desired map $\mathcal{M}_e^\alpha(G_\theta, \pm) = \mathcal{M}^\alpha(G^\theta, \pm)/\Gamma_\theta$. \square

One can define notions of simplicity for (G^θ, \pm) - and (G_θ, \pm) -Higgs bundles.

Definition 5.7. *A (G^θ, \pm) -Higgs bundle (resp. (G_θ, \pm) -Higgs bundle) (E, φ) is said to be **simple** if $\text{Aut}(E, \varphi)$ is $Z(G^\theta) \cap \ker(\iota^\pm)$ (resp. $Z(G_\theta) \cap \ker(\iota_\pm)$).*

From Proposition 4.5 a stable and simple $(G^\theta, +)$ -Higgs bundle (resp. $(G_\theta, +)$ -Higgs bundle) defines a smooth point in $\mathcal{M}(G^\theta, +)$ (resp. $\mathcal{M}(G_\theta, +)$). In contrast, for a $(G^\theta, -)$ -Higgs bundle and a $(G_\theta, -)$ -Higgs bundle to define a smooth point, in addition to stability and simplicity, the vanishing of a certain obstruction living in the second degree hypercohomology group of the deformation complex defined by the Higgs bundle (see [16]).

5.2. (G^θ, \pm) - and (G_θ, \pm) -Higgs bundles versus G -Higgs bundles. Let (E, φ) be a (G^θ, \pm) -Higgs bundle. By extending the structure group to G we obtain a G -bundle E_G and since $E_G(\mathfrak{g}) = E(\mathfrak{g}^+) \oplus E(\mathfrak{g}^-)$, we can associate to the Higgs field φ a section φ_G of $E_G(\mathfrak{g}) \otimes K$ by taking 0 in $E(\mathfrak{g}^\mp)$ component. The G -Higgs bundle (E_G, φ_G) will be referred as the **extension** of (E, φ) . Also, if (E, φ) is a G -Higgs bundle we say that it reduces to a

(G^θ, \pm) -Higgs bundle if E reduces to a G^θ -bundle E_{G^θ} and φ takes values in $E_{G^\theta}(\mathfrak{g}^\pm) \otimes K$, in which case we rename it φ^\pm . We say that $(E_{G^\theta}, \varphi^\pm)$ is a **reduction** of (E, φ) . We consider the similar construction also for (G_θ, \pm) -Higgs bundles. Recall that we say that a (G^θ, \pm) -Higgs bundle ((G_θ, \pm) -Higgs bundle) is polystable when it is 0-polystable.

Proposition 5.8. (1) *Let (E, φ) be a polystable (G^θ, \pm) -Higgs bundle. Then the corresponding G -Higgs bundles (E_G, φ_G) is also polystable. Hence the map $(E, \varphi) \mapsto (E_G, \varphi_G)$ defines a $|\Gamma_\theta| : 1$ map $\mathcal{M}(G^\theta, \pm) \rightarrow \mathcal{M}(G)$, where $|\Gamma_\theta|$ is the order of Γ_θ .*

(2) *Let (E, φ) be a G -Higgs bundle which reduces to a (G^θ, \pm) -Higgs bundle $(E_{G^\theta}, \varphi_\pm)$. Then if (E, φ) is (semi,poly)stable, $(E_{G^\theta}, \varphi_\pm)$ is also (semi,poly)polystable.*

(3) *Let $\theta, \theta' \in \text{Aut}_2(G)$ such that $\theta' = \text{Int}(g)\theta \text{Int}(g^{-1})$, with $g \in G$. Then $\text{Int}(g)$ gives rise to a canonical isomorphism of $\mathcal{M}(G^\theta, \pm)$ with $\mathcal{M}(G^{\theta'}, \pm)$. Since the action of $\text{Int}(g)$ in $\mathcal{M}(G)$ is trivial we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{M}(G^\theta, \pm) & \rightarrow & \mathcal{M}(G) \\ \downarrow & \nearrow & \\ \mathcal{M}(G^{\theta'}, \pm) & & \end{array}$$

Proof. That the polystability of a (G^θ, \pm) -Higgs bundle (E, φ) implies that of $(E_{G^\theta}, \varphi_\pm)$ follows from Theorems 4.6 and 5.5, together with the observation that $E(G^\theta/U^\sigma) \subset E_G(G/U)$ and hence a reduction of structure group of E to U^σ defines a reduction of structure group of E_G to U . The fact that the former satisfies the Hitchin equation for $\alpha = 0$ implies that the latter satisfies the Hitchin equation in Theorem 4.6 implying the polystability of (E_G, φ_G) .

The map $\mathcal{M}(G_\theta, \pm) \rightarrow \mathcal{M}(G)$ is $|\Gamma_\theta| : 1$ since G_θ is the normalizer in G of G^θ as shown in Proposition 2.19.

To prove (2) suppose that $(E_{G^\theta}, \varphi_\pm)$ is not semistable. Following Definition 5.1, there is an $s \in \mathfrak{u}^\sigma$ defining a parabolic subgroup $P_s \in G^\theta$, and a reduction σ of E_{G^θ} to a P_s -bundle such that $\deg(E_{G^\theta})(s, \sigma) < 0$. But s defines also a parabolic subgroup \tilde{P}_s of G , and the reduction σ defines a reduction $\tilde{\sigma}$ of E to \tilde{P}_s such that $\deg(E)(s, \tilde{\sigma}) = \deg(E_{G^\theta})(s, \sigma)$. The result follows now. The same argument applies to stability and polystability.

(3) follows from the fact that $\text{Int}(G)$ acts trivially on $\mathcal{M}(G)$ as explained in Section 3. \square

Similarly to Proposition 5.8, we have the following.

Proposition 5.9. (1) *Let (E, φ) be a polystable (G_θ, \pm) -Higgs bundle. Then the corresponding G -Higgs bundles (E_G, φ_G) is also polystable. Hence the map $(E, \varphi) \mapsto (E_G, \varphi_G)$ defines an injective map $\mathcal{M}(G_\theta, \pm) \rightarrow \mathcal{M}(G)$.*

(2) *Let (E, φ) be a G -Higgs bundle which reduces to a (G_θ, \pm) -Higgs bundle $(E_{G_\theta}, \varphi_\pm)$. Then if (E, φ) is (semi,poly)stable, $(E_{G_\theta}, \varphi_\pm)$ is also (semi,poly)polystable.*

(3) *Let $\theta, \theta' \in \text{Aut}_2(G)$ such that $\theta' = \text{Int}(g)\theta \text{Int}(g^{-1})$, with $g \in G$. Then $\text{Int}(g)$ gives rise to a canonical isomorphism of $\mathcal{M}(G_\theta, \pm)$ with $\mathcal{M}(G_{\theta'}, \pm)$. Since the action of $\text{Int}(g)$ in $\mathcal{M}(G)$ is trivial we have a commutative diagram*

$$\begin{array}{ccc}
\mathcal{M}(G_\theta, \pm) & \rightarrow & \mathcal{M}(G) \\
\downarrow & \nearrow & \\
\mathcal{M}(G_{\theta'}, \pm) & &
\end{array}$$

Proof. The proof is analogous to that of Proposition 5.8. We only remark that now the map $\mathcal{M}(G_\theta, \pm) \rightarrow \mathcal{M}(G)$ is injective since indeed G_θ is the normalizer of G^θ in G (see also Proposition 5.6). \square

Remark 5.10. (1) Note that the relation in Propositions 5.8 and 5.9 between (G^θ, \pm) -Higgs bundles (resp. (G_θ, \pm) -Higgs bundles) and G -Higgs bundles applies when the stability parameter α for the (G^θ, \pm) -Higgs bundles (resp. (G_θ, \pm) -Higgs bundles) is 0. Although the other values of α do not relate in general to polystable G -Higgs bundles, they turn out to play an important role in the study of the topology of the moduli space for $\alpha = 0$.

(2) In the more general case in which G is reductive, there is an analogous result to Propositions 5.8 and 5.9, for α -polystable objects, where the element α in the centre of \mathfrak{g} is determined by the topological class of the G -bundle.

5.3. Higgs bundles defined by order n automorphisms of G . Let $\theta \in \text{Aut}(G)$ of order n . We can generalise many of the constructions of the previous sections when $n = 2$ to the more general situation.

Let $0 \leq k \leq n - 1$ and $\zeta_k := \exp(2\pi i \frac{k}{n})$. A (G^θ, ζ_k) -**Higgs bundle** over X is a pair (E, φ) where E is a principal G^θ -bundle over X and φ is a section of $E(\mathfrak{g}^k) \otimes K$, where $E(\mathfrak{g}^k)$ is the bundle associated to E via the representation $\iota^k : G_\theta \rightarrow \text{GL}(\mathfrak{g}^k)$ defined in Proposition 2.27, and, as above, K is the canonical bundle on X . We can also define a (G_θ, ζ_k) -**Higgs bundle** over X as a pair (E, φ) where E is a principal G_θ -bundle over X and φ is a section of $E(\mathfrak{g}^k) \otimes K$, where $E(\mathfrak{g}^k)$ is the bundle associated to E via the representation $\iota_k : G^\theta \rightarrow \text{GL}(\mathfrak{g}^k)$ defined in Proposition 2.27.

The stability conditions given in the $n = 2$ case in Definition 5.1 can be extended in a straightforward manner to this situation, replacing \mathfrak{u}^σ by the Lie algebra of the maximal compact subgroup of G^θ , and \mathfrak{g}_s^\pm and $\mathfrak{g}_{s,0}^\pm$ by the obvious generalisations of (5.2) and (5.3) to \mathfrak{g}^k (see [16]). We thus have moduli spaces of α -polystable objects that we will denote $\mathcal{M}^\alpha(G^\theta, \zeta_k)$ and $\mathcal{M}^\alpha(G_\theta, \zeta_k)$, or simply $\mathcal{M}(G^\theta, \zeta_k)$ and $\mathcal{M}(G_\theta, \zeta_k)$ if $\alpha = 0$. There are suitable generalisations of Theorems 5.4 and 5.5 to this situation for the existence of a reduction of structure group to a maximal compact subgroup of G^θ and G_θ respectively, satisfying appropriate Hitchin equations (see [16]). We also have straightforward generalisations of Propositions 5.8 and 5.9 regarding maps $\mathcal{M}(G^\theta, \zeta_k) \rightarrow \mathcal{M}(G)$, and $\mathcal{M}(G_\theta, \zeta_k) \rightarrow \mathcal{M}(G)$, etc.

6. HIGGS BUNDLES AND REPRESENTATIONS OF THE FUNDAMENTAL GROUP

6.1. Representations of the fundamental group and harmonic reductions. In this section we take G to be a reductive Lie group (real or complex). By a **representation** of $\pi_1(X)$ in G we understand a homomorphism $\rho : \pi_1(X) \rightarrow G$. The set of all such

homomorphisms, $\text{Hom}(\pi_1(X), G)$, is an analytic variety, which is algebraic if G is algebraic. The group G acts on $\text{Hom}(\pi_1(X), G)$ by conjugation:

$$(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1}$$

for $g \in G$, $\rho \in \text{Hom}(\pi_1(X), G)$ and $\gamma \in \pi_1(X)$. If we restrict the action to the subspace $\text{Hom}^+(\pi_1(X), g)$ consisting of reductive representations, the orbit space is Hausdorff. By a **reductive representation** we mean one that, composed with the adjoint representation in the Lie algebra of G , decomposes as a sum of irreducible representations. If G is algebraic this is equivalent to the Zariski closure of the image of $\pi_1(X)$ in G being a reductive group. (When G is compact every representation is reductive.) The **moduli space of reductive representations** of $\pi_1(X)$ in G is defined to be the orbit space

$$\mathcal{R}(G) = \text{Hom}^+(\pi_1(X), G)/G.$$

It has the structure of an analytic variety (see e.g. [21]) which is algebraic if G is algebraic and is real if G is real or complex if G is complex.

Let $\rho : \pi_1(X) \rightarrow G$ be a representation of $\pi_1(X)$ in G . Let $Z_G(\rho)$ be the centralizer in G of $\rho(\pi_1(X))$. We say that ρ is **irreducible** if and only if it is reductive and $Z_G(\rho) = Z(G)$, where $Z(G)$ is the centre of G .

Given a representation $\rho : \pi_1(X) \rightarrow G$, there is an associated flat principal G -bundle on X , defined as

$$E_\rho = \tilde{X} \times_\rho G,$$

where $\tilde{X} \rightarrow X$ is the universal cover associated to X and $\pi_1(X)$ acts on G via ρ . This gives in fact an identification between the set of equivalence classes of representations $\text{Hom}(\pi_1(X), G)/G$ and the set of equivalence classes of flat principal G -bundles, which in turn is parametrized by the (nonabelian) cohomology set $H^1(X, G)$. We have the following.

An important result is the following theorem of Corlette [11], also proved by Donaldson [13] when $G = \text{SL}(2, \mathbb{C})$ (see also [29]).

Theorem 6.1. *Let G be a reductive Lie group. Let ρ be a representation of $\pi_1(X)$ in G with corresponding flat G -bundle E_ρ . Let $H \subset G$ be a maximal compact subgroup, and let $E_\rho(G/H)$ be the associated G/H -bundle. Then the existence of a harmonic section of $E_\rho(G/H)$ is equivalent to the reductivity of ρ .*

6.2. G -Higgs bundles and representations. In this section G will be a complex semisimple Lie group. We explain now the important relation between G -Higgs bundles and representations of the fundamental group in G . Let (E, φ) be a G -Higgs bundle, and let $U \subset G$ be a maximal compact subgroup of G . Let h be a reduction of structure group of E from G to U . Let d_h be the Chern connection — the unique connection on E compatible with h and the holomorphic structure of E — and let F_h be its curvature. Theorem 4.6 states that the polystability of (E, φ) is equivalent to the existence of a reduction h satisfying the Hitchin equation. A computation shows that if that is the case

$$D = d_h + \varphi - \tau_h(\varphi)$$

defines a flat connection on the principal G -bundle E , and the reduction h is harmonic. Here τ_h is as defined in Section 4.2. Hence the holonomy of this connection defines a

representation of $\pi_1(X)$ in G , which, by Theorem 6.1, is reductive. In fact all reductive representations of $\pi_1(X)$ in G arise in this way. More precisely, we have the following.

Theorem 6.2. *Let G be complex semisimple Lie group. The moduli space $\mathcal{M}(G)$ of polystable G -Higgs bundles and the moduli space $\mathcal{R}(G)$ reductive representations of $\pi_1(X)$ in G are homeomorphic. Under this homeomorphism the irreducible representations of $\pi_1(X)$ in G are in correspondence with the stable and simple G -Higgs bundles.*

Remark 6.3. If G is reductive (but not semisimple) the same result is true if we require that the topological class of the G bundle E (given by an element in $\pi_1(G)$) be trivial. If we do not make any restriction on the topological class of E there is a similar correspondence involving representations of the universal central extension of the fundamental group.

One way to understand the non-abelian Hodge theory correspondence given by Theorem 6.2 is through the analysis of the hyperkähler structure of the moduli spaces involved. In Section 4.3 we saw how $\mathcal{M}(G)$ had a hyperkähler structure. Using the same notation of Section 4.3, let us now see how the smooth locus of the moduli of harmonic flat connections on \mathbf{E}_U can be realized as a hyperkähler quotient. Let \mathcal{D} be the set of G -connections on \mathbf{E}_G . This is an affine space modelled on $\Omega^1(X, \mathbf{E}_G(\mathfrak{g})) = \Omega^0(X, T^*X \otimes_{\mathbb{R}} \mathbf{E}_G(\mathfrak{g}))$. The space \mathcal{D} has a complex structure $I_1 = 1 \otimes \sqrt{-1}$, which comes from the complex structure of the bundle. Using the complex structure of X we have also the complex structure $I_2 = \sqrt{-1} \otimes \tau_h$. We can finally consider the complex structure $I_3 = I_1 I_2$.

The reduction to U of the G -bundle \mathbf{E}_G together with a Riemannian metric in the conformal class of X defines a flat Riemannian metric $g_{\mathcal{D}}$ on \mathcal{D} which is Kähler for the above three complex structures. Hence $(\mathcal{D}, g_{\mathcal{D}}, I_1, I_2, I_3)$ is also a hyperkähler manifold. As in the previous case, the action of the gauge group \mathcal{U} on \mathcal{D} preserves the hyperkähler structure and there are moment maps

$$\mu_1(D) = d_A^* \psi, \quad \mu_2(D) = \text{im}(F_D), \quad \mu_3(D) = \text{Re}(F_D),$$

where $D = d_A + \psi$ is the decomposition of D defined by

$$\mathbf{E}_G(\mathfrak{g}) = \mathbf{E}_U(\mathfrak{u}) \oplus \mathbf{E}_U(\sqrt{-1}\mathfrak{u}).$$

Hence the moduli space of solutions to the harmonicity equations given by $F_D = 0$ and $d_A^* \psi = 0$, is $\mu_*^{-1}(0)/\mathcal{U}$, where $\mu = (\mu_1, \mu_2, \mu_3)$. If we consider the set of irreducible connections $\mu_*^{-1}(0)$, then

$$\mu_*^{-1}(0)/\mathcal{U},$$

is a hyperkähler manifold. The homeomorphism between the moduli spaces of solutions to the Hitchin and the harmonicity equations is induced from the affine map

$$\begin{aligned} \mathcal{A} \times \Omega &\longrightarrow \mathcal{D} \\ (d_A, \varphi) &\longmapsto d_A + \varphi - \tau_h(\varphi). \end{aligned}$$

One can see easily, for example, that this map sends $\mathcal{A} \times \Omega$ with complex structure J_2 to \mathcal{D} with complex structure I_1 (see [24]).

Now, Theorems 4.8 and 6.1 can be regarded as existence theorems, establishing the non-emptiness of the hyperkähler quotient, obtained by focusing on different complex structures. For Theorem 4.8 one gives a special status to the complex structure J_1 . Combining

the symplectic forms determined by J_2 and J_3 one has the J_1 -holomorphic symplectic form $\Omega_1 = \omega_2 + i\omega_3$ on $\mathcal{A} \times \Omega$. The gauge group $\mathcal{G} = \mathcal{U}^C$ acts on $\mathcal{A} \times \Omega$ preserving Ω_1 . The symplectic quotient construction can also be extended to the holomorphic situation (see e.g. [27]) to obtain the holomorphic symplectic quotient $\{(\bar{\partial}_E, \varphi) : \bar{\partial}_E \varphi = 0\}/\mathcal{G}$. What Theorem 4.8 says is that for a class $[(\bar{\partial}_E, \varphi)]$ in this quotient to have a representative (unique up to \mathcal{U} -gauge transformations) satisfying $\mu_1 = 0$ it is necessary and sufficient that the pair $(\bar{\partial}_E, \varphi)$ be polystable. This identifies the hyperkähler quotient to the set of equivalence classes of polystable G -Higgs bundles on \mathbf{E}_G . If one now takes J_2 on $\mathcal{A} \times \Omega$ or equivalently \mathcal{D} with I_1 and argues in a similar way, one gets Theorem 6.1 identifying the hyperkähler quotient to the set of equivalence classes of reductive flat connections on \mathbf{E}_G .

6.3. (G^θ, \pm) - and (G_θ, \pm) -Higgs bundles and representations. Let G be a complex semisimple Lie group and let $\theta \in \text{Aut}_2(G)$. Recall from Section 2.3 that we can choose a compact conjugation τ of G , defining a compact real form U of G such that $\tau\theta = \theta\tau$. We then consider the complex conjugation σ of G defined by $\sigma = \theta\tau$. This defines a real form G^σ of G . Consider the notations used in Sections 2.4 and 5.1.

Applying similar arguments to the ones used to prove Theorem 6.2, we can combine Theorem 6.1 with Theorems 5.4 and 5.5 to prove the following (see [16] for details).

Theorem 6.4. *Let G be a complex semisimple Lie group and let $\theta \in \text{Aut}_2(G)$. Let τ be a compact conjugation of G commuting with θ and σ be the conjugation of G defined by $\sigma := \theta\tau$. We have the following:*

- (1) $\mathcal{M}(G^\theta, +)$ is homeomorphic to $\mathcal{R}(G^\theta)$,
- (2) $\mathcal{M}(G^\theta, -)$ is homeomorphic to $\mathcal{R}(G^\sigma)$,
- (3) $\mathcal{M}(G_\theta, +)$ is homeomorphic to $\mathcal{R}(G_\theta)$,
- (4) $\mathcal{M}(G_\theta, -)$ is homeomorphic to $\mathcal{R}(G_\sigma)$.

Under these homeomorphisms the irreducible representations are in correspondence with the stable and simple objects.

Remark 6.5. In contrast with the order 2 case, when $\theta \in \text{Aut}_n(G)$, the (G^θ, ζ_k) - and (G_θ, ζ_k) -Higgs bundles do not have in general an interpretation in terms of representations of the fundamental group of X , except when $k = 0$ since a (G_θ, ζ_0) -Higgs bundle (resp. (G_θ, ζ_0) -Higgs bundle) is simply a G^θ -Higgs bundle (resp. G_θ -Higgs bundle) and hence corresponds with a representation of $\pi_1(X)$ in G^θ (resp. in G_θ). They are related however to so-called Hodge bundles and variations of Hodge structure.

7. INVOLUTIONS OF $\mathcal{M}(G)$

In this section G is a connected complex semisimple Lie group, X is a compact Riemann surface, and we consider the moduli space $\mathcal{M}(G)$ of G -Higgs bundles over X . Our goal is to study fixed points of various involutions of $\mathcal{M}(G)$. These involve the moduli spaces $\mathcal{M}(G^\theta, \pm)$ and $\mathcal{M}(G_\theta, \pm)$ defined in Section 5, where $\theta \in \text{Aut}_2(G)$, and G^θ and G_θ are the subgroups of G defined in Section 2.4. We will denote by $\mathcal{M}(G)_{ss}$ the subvariety of stable and simple points of $\mathcal{M}(G)$ and by $\widetilde{\mathcal{M}}(G^\theta, \pm)$ and $\widetilde{\mathcal{M}}(G_\theta, \pm)$ the images of $\mathcal{M}(G^\theta, \pm)$ and $\mathcal{M}(G_\theta, \pm)$, respectively in $\mathcal{M}(G)$ under the maps defined in Propositions 5.8 and 5.9 respectively.

7.1. The involution $(E, \varphi) \mapsto (E, -\varphi)$.

Proposition 7.1. *Let (E, φ) be a simple G -Higgs bundle isomorphic to $(E, -\varphi)$. Then:*

(1) *The structure group of E can be reduced to the centralizer of an element $s \in G$ such that s^2 belongs to the centre, and the element s is unique up to the right action of G by inner automorphisms and the action of $Z(G)$ by multiplication.*

(2) *The Higgs field φ takes values in the (-1) -eigenspace of the automorphism of \mathfrak{g} defined by $\text{Int}(s)$. In other words, (E, φ) reduces to a $(G^\theta, -)$ -Higgs bundle, where $\theta = \text{Int}(s)$.*

Proof. Suppose that A is an automorphism of E such that $\text{Ad}(A)(\varphi) = -\varphi$, where $\text{Ad}(A)$ is the automorphism of $E(\mathfrak{g}) \otimes K$ induced by A . Then A^2 is an automorphism of (E, φ) , and since (E, φ) is simple (recall Definition 4.3), $A^2 = z$ with $z \in Z$. By Proposition 3.3 this defines a unique G -orbit in the set $S_{\text{Id}} = \{s \in G : s^2 = z \in Z\}$. The structure group of E can be reduced to $Z_G(s)$ for any s in this orbit. If A' is another automorphism of E such that $\text{Ad}(A')(\varphi) = -\varphi'$, then $A' = z'A$, for $z' \in Z$. This follows from the simplicity of (E, φ) since AA' is an automorphism of (E, φ) . Clearly if $f_A(\xi) = s$ we have $f_{A'}(\xi) = z's$. This concludes the proof of (1).

To prove (2), let $\theta := \text{Int}(s)$ for an $s \in G$ as in (1). If $\varphi = 0$ then E itself is simple and $A \in Z$. Hence $\text{Ad}(A)$ is the identity. Suppose then that $\varphi \neq 0$. Then θ is non trivial and the bundle E reduces to a G^θ -bundle E_{G^θ} . The adjoint bundle decomposes as

$$E(\mathfrak{g}) = E_{G^\theta}(\mathfrak{g}^+) \oplus E_{G^\theta}(\mathfrak{g}^-),$$

where $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ is the decomposition of \mathfrak{g} in (± 1) -eigenspaces of θ (see Proposition 2.21). It is clear that $\text{Ad}(A)(\varphi) = -\varphi$ is equivalent to $\varphi \in H^0(X, E_{G^\theta}(\mathfrak{g}^-))$. \square

Remark 7.2. Notice that in fact if E is itself simple, $A \in Z$ and hence $\text{Ad}(A)$ is the identity, implying that $\varphi = 0$.

Recall the quotient $\text{Int}_2(G)/\sim$ defined in Section 2.2. We have the following.

Theorem 7.3. *Consider the involution*

$$\begin{aligned} \iota : \mathcal{M}(G) &\rightarrow \mathcal{M}(G) \\ (E, \varphi) &\mapsto (E, -\varphi). \end{aligned}$$

Then

(1)

$$\bigcup_{[\theta] \in \text{Int}_2(G)/\sim} \widetilde{\mathcal{M}}(G^\theta, -) \subset \mathcal{M}(G)^\iota,$$

(2)

$$\mathcal{M}(G)_{ss}^\iota \subset \bigcup_{[\theta] \in \text{Int}_2(G)/\sim} \widetilde{\mathcal{M}}(G^\theta, -).$$

Proof. Let $\theta \in \text{Int}_2(G)$, and $(E, \varphi) \in \mathcal{M}(G^\theta, -)$. From (1) in Proposition 5.8 the image of (E, φ) in $\mathcal{M}(G)$ is given by the extended G -Higgs bundle (E_G, φ_G) . Of course $\theta(E_G) \cong E_G$ since $\theta \in \text{Int}(G)$, and clearly $\theta(\varphi_G) = -\varphi_G$. Hence θ defines an isomorphism between (E_G, φ_G) and $(E_G, -\varphi_G)$, showing that $(E_G, \varphi_G) \in \mathcal{M}(G)^\iota$. To complete the proof of (1),

we apply (3) in Proposition 5.8, which says that if $\theta \sim \theta'$, $\mathcal{M}(G^\theta, -)$ and $\mathcal{M}(G^{\theta'}, -)$ are isomorphic and their images in $\mathcal{M}(G)$ coincide.

The proof of (2) follows from Proposition 7.1 combined with (2) and (3) of Proposition 5.8 and (4) of Proposition 2.7. \square

7.2. Outer involutions of $\mathcal{M}(G)$. If $a \in \text{Out}_2(G)$ and θ is a lift of a to $\text{Aut}_2(G)$, we would like to know when $\theta(E, \varphi) \simeq (E, \pm\varphi)$. If $\theta \in \text{Int}_2(G)$ then the isomorphism of E with $\theta(E)$ takes φ to itself. We will therefore assume that $a \neq 1$. We have the following.

Proposition 7.4. *Let $\theta \in \text{Aut}_2(G)$, and let (E, φ) be a (G^θ, \pm) -Higgs bundle and (E_G, φ_G) be the corresponding extension to a G -Higgs bundle. Then (E_G, φ_G) is isomorphic to $(\theta(E_G), \pm\theta(\varphi_G))$.*

Proof. The bundle E_G is obtained from E by the extension $G^\theta \subset G$ of structure group. The extension obtained by further composition with θ gives $\theta(E_G)$. Since θ is identity on G^θ there is a canonical isomorphism of E_G with $\theta(E_G)$. If φ takes values in the ± 1 -eigenspace of $\text{ad}(\theta)$, it gives rise to a Higgs field on E_G on which $\text{ad}(\theta)$ acts as ± 1 . \square

The following generalises Proposition 7.1.

Proposition 7.5. *Let $\theta \in \text{Aut}_2(G)$, and let (E, φ) be a simple G -Higgs bundle isomorphic to $(\theta(E), \pm\theta(\varphi))$. Then we have the following.*

(1) *The structure group of E can be reduced to $G^{\theta'}$ with $\theta' = \text{Int}(s)\theta$ and $s \in S_\theta$, where S_θ as defined in Proposition 2.7. Moreover s is unique up to the action of G and $Z = Z(G)$ defined in Proposition 2.7.*

(2) *The Higgs field φ takes values in the ± 1 -eigenspace of the automorphism of \mathfrak{g} defined by θ' . In other words, (E, φ) reduces to a $(G^{\theta'}, \pm)$ -Higgs bundle.*

Proof. Suppose that (E, φ) is isomorphic to $(\theta(E), \pm\theta(\varphi))$. This means that there is an isomorphism of $A : E \rightarrow \theta(E)$ such that $\text{Ad}(A)(\varphi) = \pm\theta(\varphi)$. From Proposition 3.4, the isomorphism A can be identified with a θ -twisted automorphism of E , as defined in Section 3.2. Since θ is of order 2, A^2 defines an automorphism of (E, φ) , and hence $A^2 = z$ with $z \in Z$ since (E, φ) is simple. From Proposition 3.9 the function f_A given in (3.1) maps E onto a single orbit of the set S_θ^2 defined in Proposition 2.7 under the action of G defined there. If $A' : E \rightarrow \theta(E)$ is another isomorphism such that $\text{Ad}(A')(\varphi) = \pm\theta(\varphi)$ then $AA' = z'$ for some $z' \in Z$ and the orbit defined by A' is given by multiplication by z' . We thus obtain a unique orbit in S_θ^2 under the action of $G \times Z$.

Now, by (2) in Proposition 3.9, every element s in this $Z \times G$ -orbit defines a reduction of structure group of E to $G^{\theta'}$, where $\theta' = \text{Int}(s)\theta$, and $G^{\theta'}$ is the subgroup of G of fixed points under θ' . This concludes the proof of (1).

To prove (2), let $\theta' := \text{Int}(s)\theta$ for $s \in S_\theta^2$ in the $Z \times G$ -orbit defined in (1). The bundle E reduces then to a $G^{\theta'}$ -bundle $E_{G^{\theta'}}$ and the adjoint bundle decomposes as

$$E(\mathfrak{g}) = E_{G^{\theta'}}(\mathfrak{g}^+) \oplus E_{G^{\theta'}}(\mathfrak{g}^-),$$

where $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ is the decomposition of \mathfrak{g} in (± 1) -eigenspaces of θ (see Proposition 2.21). It is clear that $\text{Ad}(A)(\varphi) = \pm\theta(\varphi)$ is equivalent to $\varphi \in H^0(X, E_{G^{\theta'}}(\mathfrak{g}^\pm))$. \square

Theorem 7.6. *Let $a \in \text{Out}_2(G)$. Consider the involutions*

$$\begin{aligned} \iota(a, \pm) : \mathcal{M}(G) &\rightarrow \mathcal{M}(G) \\ (E, \varphi) &\mapsto (a(E), \pm a(\varphi)). \end{aligned}$$

Then

(1)

$$\bigcup_{[\theta] \in \mathcal{C}^{-1}(a)} \widetilde{\mathcal{M}}(G^\theta, \pm) \subset \mathcal{M}(G)^{\iota(a, \pm)},$$

(2)

$$\mathcal{M}(G)_{ss}^{\iota(a, \pm)} \subset \bigcup_{[\theta] \in \mathcal{C}^{-1}(a)} \widetilde{\mathcal{M}}(G^\theta, \pm)$$

(except for $\iota(1, +)$),

where $\mathcal{C} : \text{Aut}_2(G)/\sim \rightarrow \text{Out}_2(G)$ is defined in Proposition 2.6, and $\mathcal{C}^{-1}(a)$ is, by Proposition 2.9, in bijection with $H_a^1(\mathbb{Z}/2, \text{Ad}(G))$.

Proof. Let $a \in \text{Out}_2(G)$ and (E, φ) be a G -Higgs bundle. Recall from Section 3 that $(E, \varphi) \cong (a(E), \pm a(\varphi))$ is equivalent to $(E, \varphi) \cong (\theta(E), \pm \theta(\varphi))$ for any $\theta \in \text{Aut}_2(G)$, such that $\pi(\theta) = a$, where $\pi : \text{Aut}_2(G) \rightarrow \text{Out}_2(G)$ is the natural projection.

Let $\theta \in \text{Aut}_2(G)$, and $(E, \varphi) \in \mathcal{M}(G^\theta, \pm)$. From (1) in Proposition 5.8 the image of (E, φ) in $\mathcal{M}(G)$ is given by the extended G -Higgs bundle (E_G, φ_G) . From Proposition 7.4, (E_G, φ_G) is isomorphic to $(\theta(E_G), \pm \theta(\varphi_G))$, showing that $(E_G, \varphi_G) \in \mathcal{M}(G)^{\iota(a, \pm)}$. To complete the proof of (1), we apply (3) in Proposition 5.8, which says that if $\theta \sim \theta'$, $\mathcal{M}(G^\theta, \pm)$ and $\mathcal{M}(G^{\theta'}, \pm)$ are isomorphic and their images in $\mathcal{M}(G)$ coincide.

The proof of (2) follows from Proposition 7.5 combined with (2) and (3) of Proposition 5.8 and (3) of Proposition 2.7. \square

Remark 7.7. If $a = 1$, the map $\iota(a, +)$ is the identity map of $\mathcal{M}(G)$ and (1) follows trivially from Proposition 5.8.

7.3. Involutions of $\mathcal{M}(G)$ defined by elements of $H^1(X, Z) \rtimes \text{Out}(G)$. Let $(\alpha, a) \in (H^1(X, Z) \rtimes \text{Out}(G))_2$. As explained in Section 3, this means that $a \in \text{Out}_2(G)$ and $a(\alpha) = \alpha^{-1}$. With such a pair we can define involutions

$$(7.1) \quad \begin{aligned} \iota(a, \alpha, \pm) : \mathcal{M}(G) &\rightarrow \mathcal{M}(G) \\ (E, \varphi) &\mapsto (a(E) \otimes \alpha, \pm a(\varphi)). \end{aligned}$$

To describe the fixed points of these involutions, recall from Section 2.4 that given $\theta \in \text{Aut}_2(G)$ we have subgroups G^θ and G_θ of G and the exact sequence (2.9). We have the following.

Proposition 7.8. *Let $\theta \in \text{Aut}_2(G)$ and let*

$$c_\theta : H^1(X, \Gamma_\theta) \rightarrow H^1(X, Z)$$

be the map induced by the injective homomorphism $\tilde{c} : \Gamma_\theta \rightarrow Z$ defined in Proposition 2.15. Let $\gamma_\theta : H^1(X, \underline{G}_\theta) \rightarrow H^1(X, \Gamma_\theta)$ be the map defined in (5.5). We have the following:

(1) The map c_θ is injective.

(2) Let (E, φ) be a (G_θ, \pm) -Higgs bundle with $\gamma_\theta(E) = \gamma$, and let $\alpha := c_\theta(\gamma)$. Let (E_G, φ_G) be the extension of (E, φ) to a G -Higgs bundle. Then (E_G, φ_G) is isomorphic to $(\theta(E_G) \otimes \alpha, \pm\theta(\varphi_G))$.

Proof. Both Γ_θ and Z are finite groups and hence the first cohomology of X with coefficient in these groups are given by $\text{Hom}(\pi_1(X), \Gamma_\theta)$ and $\text{Hom}(\pi_1(X), Z)$ respectively. Hence c_θ is injective.

(2) Let (E, φ) be a (G_θ, \pm) -Higgs bundle. Under the natural homomorphisms $G_\theta \rightarrow \Gamma_\theta \rightarrow Z$ we get, by extension of structure group a Z -principal bundle α . On the other hand, the inclusion of G_θ in G gives rise to a G -Higgs bundle (E_G, φ_G) . Clearly, $\theta(E_G)$ is obtained from E by extension of structure group $\theta \circ \iota$ where ι is the inclusion of G_θ in G . But $\theta \circ \iota = \iota \circ \theta|_{G_\theta}$. Now, if $x \in G_\theta$, we have $\theta(x) = xx^{-1}\theta(x) = x(\tilde{c} \circ \eta)(x)$ where η is the natural surjection $G_\theta \rightarrow \Gamma_\theta$ and \tilde{c} is the inclusion of Γ_θ in Z . Now $E_G \otimes \alpha$ is obtained by extension of structure group by the multiplication map $G \times Z \rightarrow G$. Hence $\theta(E_G) \otimes \alpha$ is got from E by the extension of structure group by $g \rightarrow (\theta(g), \theta(g)^{-1}g) \rightarrow g$. This proves our assertion. \square

The following generalises Proposition 7.5.

Proposition 7.9. *Let $\theta \in \text{Aut}_2(G)$ and $\alpha \in H^1(X, Z)$ such that $\theta(\alpha) = \alpha^{-1}$, and let (E, φ) be a simple G -Higgs bundle isomorphic to $(\theta(E) \otimes \alpha, \pm\theta(\varphi))$. Then, except for the case $\theta \in \text{Int}_2(G)$, $\alpha = 1$ and (E, φ) is isomorphic to $(\theta(E), \theta(\varphi))$, which is always the case, we have the following.*

(1) The structure group of E can be reduced to $G_{\theta'}$ with $\theta' = \text{Int}(s)\theta$ and $s \in S_\theta$, where S_θ as defined in Proposition 2.7, with s unique up to the action of $Z \times G$ defined in Proposition 2.7. Moreover, if $\gamma \in H^1(X, \Gamma_{\theta'})$ is the class of the reduced $G_{\theta'}$ -bundle under the map defined in (5.5), then $c_{\theta'}(\gamma) = \alpha$, with $c_{\theta'}$ is as defined in Proposition 7.8.

(2) The Higgs field φ takes values in the ± 1 -eigenspace of the automorphism of \mathfrak{g} defined by θ' . In other words, (E, φ) reduces to a $(G_{\theta'}, \pm)$ -Higgs bundle.

This proposition is a consequence of the following more general statement.

Proposition 7.10. *Let $\theta \in \text{Aut}_2(G)$ and Γ a central subgroup of G invariant under θ . Let A be an isomorphism of the Higgs bundle (E, φ) with $(\theta(E) \otimes \alpha, \pm\theta(\varphi))$, where α is a Γ -bundle such that $\theta(\alpha) \simeq \alpha^{-1}$. Also assume that the composite of $\theta(A)$ with A gives rise to an automorphism of (E, φ) which is induced by an element of Γ . Then we have*

(1) If $t \in G$ such that $t\theta(t)$ is in the centre Z , then $\theta' = \text{Int}(t)\theta$ is also an involution of G . With our assumption above, E can be reduced to a G' -bundle E' where $G' = G_{\theta'} = \{g \in G : g\theta'(g)^{-1} \in \Gamma\}$ for some t satisfying $t\theta(t) \in \Gamma$. Let $\Gamma' = \Gamma_{\theta'}$ be the image of G' in Γ given by the homomorphism $g \rightarrow g\theta'(g)^{-1}$. Then by extension of structure group by the homomorphism $G' \rightarrow \Gamma' \subset \Gamma$ the bundle E' gives rise to a Γ' -bundle α' and a Γ -bundle which is isomorphic to α .

(2) The Higgs field φ takes values in the ± 1 -eigenspace of \mathfrak{g} defined by θ' . In other words, (E, φ) reduces to a $(G', \pm 1)$ -bundle.

(3) The isomorphism A is induced by the natural isomorphism of E' with $\theta(E') \otimes \alpha'$ as G' -bundles using multiplication by t which gives an isomorphism of $\theta(E)$ and $\theta'(E)$.

Proof. Firstly, we recall that the bundle $E \otimes \alpha$ is simply the quotient of $E \times \alpha$ by the action $\gamma(\xi, a) = (\xi\gamma, \gamma^{-1}a)$, $\gamma \in \Gamma$. The (right) action of G on $E \times \alpha$ given by $(\xi, a)s = (\xi s, a)$ goes down to an action on $E \otimes \alpha$ and makes it a principal G -bundle. The image of (ξ, a) may as well be written as $\xi \otimes a$. The projection $E \times \alpha \rightarrow E$ goes down to a map $q : E \otimes \alpha \rightarrow E/\Gamma$. Note that E/Γ is a principal G/Γ -bundle. We will denote the image of $\xi \in E$ in E/Γ by $\bar{\xi}$ and the image of $s \in G$ in G/Γ by \bar{s} . Then we have $q((\xi \otimes a)s) = q(\xi \otimes a)(\bar{s})$.

Note that as a space over X , the bundle $\theta(E)$ is the same as E except that the action of G is now $\xi \cdot s = \xi\theta(s)$. Hence $F = \theta(E) \otimes \alpha$ is the quotient of $E \times \alpha$ under the action of Γ by $\gamma(\xi, a) = (\xi\theta\gamma, \gamma^{-1}a)$. The action of G on $\theta(E) \times \alpha$ given by $(\xi, a) \cdot s = (\xi\theta(s), a)$ for $s \in G$ goes down to an action of G on F making it a principal G -bundle.

We are given an isomorphism $A : E \rightarrow \theta(E) \otimes \alpha$. Let p be the composite $q \circ A : E \rightarrow \theta(E)/\Gamma = E/\Gamma$ and let ξ be an element of E . If $A(\xi) = \eta \otimes a$, we have

$$A(\xi s) = (\eta \otimes a) \cdot s = (\eta \cdot s \otimes a) = (\eta\theta(s) \otimes a)$$

so that $p(\xi s) = p(\xi)\bar{\theta(s)}$ for all $s \in G$. We now have two (fibre-respecting) morphisms $E \rightarrow E/\Gamma$, namely, the natural quotient map π and the morphism p . Hence there exists a morphism $f_A : E \rightarrow G/\Gamma$ such that $p(\xi) = \pi(\xi)f_A(\xi)$. Also we have $\pi(\xi s) = \pi(\xi)\bar{s}$. Thus $p(\xi s) = p(\xi)\bar{\theta(s)} = \pi(\xi)f_A(\xi)\bar{\theta(s)}$ on the one hand, and $p(\xi s) = \pi(\xi s)f_A(\xi s) = \pi(\xi)\bar{s}f_A(\xi s)$ for all $s \in G$ on the other. Hence $f_A(\xi s) = \bar{s}^{-1}f_A(\xi)\bar{\theta(s)}$.

We let G act on G/Γ by $s.\bar{g} = \overline{sg\theta s^{-1}}$. The computation above shows that the composite g_A of f_A with the natural map of G/Γ onto the quotient for the above action of G , is invariant under the action of G on E . Hence it induces a morphism of X into this quotient. Note now that X is projective while this quotient is an affine variety. Hence the morphism g_A is constant.

As in Proposition 3.3, we see that the image of g_A consists of stable points for the action of G/Γ on itself. Hence there is an element $\tau \in G/\Gamma$ such that for every $\xi \in E$, we have $f_A(\xi) = \bar{g}\tau\bar{\theta(g)}^{-1}$ for some $g \in G$. Let $t \in G$ such that $\bar{t} = \tau$.

Now the isomorphism A induces an isomorphism $\theta(E) \rightarrow E \otimes \theta(\alpha) = E \otimes \alpha^{-1}$ and hence an isomorphism $\theta(A) : \theta(E) \otimes \alpha \rightarrow E$. Composing this with A we get an automorphism of (E, φ) which, by assumption, is given by an element $\gamma \in \Gamma$. Clearly this implies that $t\theta(t) \in \Gamma$, i.e. $t \in \Gamma_\theta$.

For every $\xi \in E$, there exists $a \in \alpha$ such that $A(\xi) = \xi g t \theta(g)^{-1} \otimes a$ for some $g \in G$ and $a \in \alpha$. Consider now the subspace E' of E given by

$$E' = \{\xi \in E : A(\xi) = \xi t \otimes a \text{ for some } a \in \alpha\}$$

Then for every $\xi \in E'$, we have $A(\xi s) = (\xi t \otimes a) \cdot s = \xi t \theta(s) \otimes a = \xi s.(s^{-1}t\theta(s)) \otimes a$ for any $s \in G$. In particular, if s satisfies $s^{-1}t\theta(s)t^{-1} \in \Gamma$, then (and only then) $\xi s \in E'$, that is to say, ξ and ξs are both in E' if and only if $s \in G' = G_{\text{Int}(t)\theta}$. This shows that the structure group of E can be reduced to G' and that E' provides such a reduction.

Moreover, if $\xi \in E'$ then there is a unique $a(\xi)$ such that $A(\xi) = \xi t \otimes a(\xi)$. Let $s \mapsto s\theta'(s)^{-1}$ be the homomorphism $\rho : G' \rightarrow \Gamma'$. Then the above computation shows that

$a(\xi s) = a(\xi)\rho(s)$ for all $\xi \in E'$ and $s \in \Gamma'$. The image $a(E')$ is a Γ' bundle which gives a reduction of the structure group of α to Γ' .

This proves assertion (1).

Assertion (2) is obvious.

Assertion (3). Notice that α' is given by extension of structure group by the homomorphism $g \rightarrow g\theta'(g)^{-1}$ of $G' \rightarrow \Gamma'$. Hence $\theta'(E) \otimes \alpha'$ is given by the extension $g \rightarrow g\theta'(g)^{-1}\theta'(g)$ which is the identity! So there is a natural isomorphism of E' with $\theta'(E') \otimes \alpha'$. From the definition it is clear that the extension of this natural isomorphism is A as claimed. \square

We can now state the main result of this section.

Theorem 7.11. *Let $a \in \text{Out}_2(G)$ and $\alpha \in H^1(X, Z)$ such that $a(\alpha) = \alpha^{-1}$. Consider the involutions*

$$\begin{aligned} \iota(a, \alpha, \pm) : \mathcal{M}(G) &\rightarrow \mathcal{M}(G) \\ (E, \varphi) &\mapsto (a(E) \otimes \alpha, \pm a(\varphi)). \end{aligned}$$

Then

(1)

$$\bigcup_{[\theta] \in \mathcal{C}l^{-1}(a), c_\theta(\gamma) = \alpha} \widetilde{\mathcal{M}}_\gamma(G_\theta, \pm) \subset \mathcal{M}(G)^{\iota(a, \alpha, \pm)}.$$

(2)

$$\mathcal{M}(G)_{ss}^{\iota(a, \alpha, \pm)} \subset \bigcup_{[\theta] \in \mathcal{C}l^{-1}(a), c_\theta(\gamma) = \alpha} \widetilde{\mathcal{M}}_\gamma(G_\theta, \pm)$$

(except for $\iota(1, 1, +)$),

where $\mathcal{C}l : \text{Aut}_2(G)/\sim \rightarrow \text{Out}_2(G)$ is defined in Proposition 2.6, and $\mathcal{C}l^{-1}(a)$ is, by Proposition 2.9, in bijection with $H_a^1(\mathbb{Z}/2, \text{Ad}(G))$.

Proof. Let $a \in \text{Out}_2(G)$ and $\alpha \in H^1(X, Z)$ such that $a(\alpha) = \alpha^{-1}$, and let (E, φ) be a G -Higgs bundle. Recall from Section 3 that $(E, \varphi) \cong (a(E) \otimes \alpha, \pm a(\varphi))$ is equivalent to $(E, \varphi) \cong (\theta(E) \otimes \alpha, \pm \theta(\varphi))$ for any $\theta \in \text{Aut}_2(G)$, such that $\pi(\theta) = a$, where $\pi : \text{Aut}_2(G) \rightarrow \text{Out}_2(G)$ is the natural projection.

Let $\theta \in \text{Aut}_2(G)$ and let now (E, φ) be an element in $\mathcal{M}_\gamma(G_\theta, \pm)$. From (1) in Proposition 5.9 the image of (E, φ) in $\mathcal{M}(G)$ is given by the extended G -Higgs bundle (E_G, φ_G) . From Proposition 7.8 (E_G, φ_G) is isomorphic to $(\theta(E_G) \otimes \alpha, \pm \theta(\varphi_G))$, with $\alpha = c_\theta(\gamma)$. showing that $(E_G, \varphi_G) \in \mathcal{M}(G)^{\iota(a, \alpha, \pm)}$ if $\alpha = c_\theta(\gamma)$. To complete the proof of (1), we apply (3) in Proposition 5.9, which says that if $\theta \sim \theta'$, $\mathcal{M}_\gamma(G_\theta, \pm)$ and $\mathcal{M}_\gamma(G_{\theta'}, \pm)$ are isomorphic and their images in $\mathcal{M}(G)$ coincide.

The proof of (2) follows from Proposition 7.9 combined with (2) and (3) of Proposition 5.9 and (3) of Proposition 2.7. \square

Remark 7.12. If $a = 1$ and $\alpha = 1$, the map $\iota(a, \alpha, +)$ is the identity map of $\mathcal{M}(G)$ and (1) follows trivially from Proposition 5.9.

Remark 7.13. It is clear that Theorem 7.6 is a special case of Theorem 7.11, obtained by taking $\alpha \in H^1(X, Z)$ to be the neutral element. Note that for the neutral element $e \in H^1(X, \Gamma_\theta)$, from Proposition 5.6, we have $\mathcal{M}_e(G_\theta, \pm) = \mathcal{M}(G^\theta, \pm)/\Gamma_\theta$, thus implying from Propositions 5.8 and 5.9, that $\widetilde{\mathcal{M}}_e(G_\theta, \pm) = \widetilde{\mathcal{M}}(G^\theta, \pm)$.

Remark 7.14. It is important to point out that the second statement in Theorems 7.3, 7.6 and 7.11 cannot be extended in general to the whole moduli space $\mathcal{M}(G)$. The fixed points of the singular locus in $\mathcal{M}(G)$ may lead to extra components that are not in general (G_θ, \pm) -Higgs bundles. Indeed, a strictly polystable G -Higgs bundle can reduce its structure group to a reductive subgroup $H \subset G$ for which the resulting H -Higgs bundle is both stable and simple (see [18]). One has to analyse the precise relation between the cliques of H and G to give a description of the fixed points. This issue arises already for $G = \mathrm{SL}(n, \mathbb{C})$ as we will see in Section 9.

Remark 7.15. As mentioned in Section 4.2, fixing the topological class c of E we can consider $\mathcal{M}_c(G) \subset \mathcal{M}(G)$, the moduli space of semistable G -Higgs bundles with fixed topological class c . This is connected and non-empty (see [18]). If G is connected the topological class is an element of $c \in \pi_1(G)$. Of course $\mathrm{Out}(G)$ acts on $\pi_1(G)$, and we require for fixed points of the involutions studied above to exist that c be fixed under the element $a \in \mathrm{Out}_2(G)$. Recall that, since G is semisimple, Z is finite and hence the topological class of a G -bundle E coincides with that of $E \otimes \alpha$ for any $\alpha \in H^1(X, Z)$.

8. INVOLUTIONS OF THE MODULI SPACE OF REPRESENTATIONS

The involutions of $\mathcal{M}(G)$ studied in Section 7 induce naturally involutions on the moduli space of representations $\mathcal{R}(G)$, under the homeomorphism between $\mathcal{M}(G)$ and $\mathcal{R}(G)$ given by Theorem 6.2. We analyse now these involutions.

8.1. Action on representations. Let (E, φ) be a polystable G -Higgs bundle and let h be a solution to the Hitchin equation given by Theorem 4.6. Recall from Section 6.2 that

$$D = d_h + \varphi - \tau_h(\varphi)$$

defines a flat connection on the principal G -bundle E , where d_h is the unique connection on E compatible with h and the holomorphic structure of E , and τ_h is defined, by h , a conjugation τ of G defining a compact real form, and the natural conjugation on $(1, 0)$ -forms on X (see (4.5) for the precise definition).

Let $\theta \in \mathrm{Aut}_2(G)$. From Proposition 2.2, we can choose θ in the class in $\mathrm{Aut}_2(G)/\sim$ such that $\theta\tau = \tau\theta$. Let σ be the conjugation of G defined by $\sigma := \theta\tau$, and $\sigma_h := \theta\tau_h$. We have the following

Proposition 8.1. (1) *The flat G -connection corresponding to $(\theta(E), \theta(\varphi))$ is given by $\theta(D)$.*

(2) *The flat G -connection corresponding to $(\theta(E), -\theta(\varphi))$ is given by $\sigma_h(D)$.*

Proof. Let us represent the holomorphic structure of E by a Dolbeault operator $\bar{\partial}_E$. We then have that $d_h = \bar{\partial}_E + \tau_h(\bar{\partial}_E)$. From this we have

$$\theta(D) = \theta(\bar{\partial}_E) + \theta\tau_h(\bar{\partial}_E) + \theta(\varphi) - \theta\tau_h(\varphi).$$

But $\theta\tau_h = \tau_h\theta$ and $\theta(\bar{\partial}_E) = \bar{\partial}_{\theta(E)}$, proving (1).

The proof of (2) follows from the following computation:

$$\begin{aligned}\sigma_h(D) &= \sigma_h(\bar{\partial}_E) + \sigma_h\tau_h(\bar{\partial}_E) + \sigma_h(\varphi) - \sigma_h\tau_h(\varphi) \\ &= \tau_h\theta(\bar{\partial}_E) + \theta(\bar{\partial}_E) + \tau_h\theta(\varphi) - \theta(\varphi) \\ &= \bar{\partial}_{\theta(E)} + \tau_h(\bar{\partial}_{\theta(E)}) - \theta(\varphi) - \tau_h(-\theta(\varphi)).\end{aligned}$$

□

From this we immediately have the following.

Proposition 8.2. *Let $\theta \in \text{Aut}_2(G)$, and $\sigma = \theta\tau$. Let $\alpha \in H^1(X, Z)$ and $\lambda : \pi_1(X) \rightarrow Z$ its corresponding representation such that $\theta(\alpha) = \alpha^{-1}$, which is equivalent to $\theta(\lambda) = \sigma(\lambda) = \lambda^{-1}$. Let (E, φ) be a polystable G -Higgs bundle and ρ be the corresponding element in $\mathcal{R}(G)$. Then:*

(1) *The involution of $\mathcal{M}(G)$ defined by $(E, \varphi) \mapsto (\theta(E) \otimes \alpha, \theta(\varphi))$ corresponds with the involution of $\mathcal{R}(G)$ given by $\rho \mapsto \lambda\theta(\rho)$.*

(2) *The involution of $\mathcal{M}(G)$ defined by $(E, \varphi) \mapsto (\theta(E) \otimes \alpha, -\theta(\varphi))$ corresponds with the involution of $\mathcal{R}(G)$ given by $\rho \mapsto \lambda\sigma(\rho)$.*

The involutions $\rho \mapsto \lambda\theta(\rho)$ and $\rho \mapsto \lambda\sigma(\rho)$ depend naturally only on the cliques $\mathcal{C}([\theta])$ and $\widehat{\mathcal{C}}([\sigma])$, respectively, where \mathcal{C} and $\widehat{\mathcal{C}}$ are defined in Section 2.3. Let $a \in \text{Out}_2(G)$ and $\theta \in \text{Aut}_2(G)$ such that $\mathcal{C}([\theta]) = a$. Let $\sigma = \theta\tau$ be the corresponding conjugation. Of course $\widehat{\mathcal{C}}([\sigma]) = a$. Let $\rho \in \mathcal{R}(G)$. We define

$$a^+(\rho) = \theta(\rho) \quad \text{and} \quad a^-(\rho) = \sigma(\rho).$$

Recall from Theorem 6.2 that the smooth locus $\mathcal{R}(G)_i \subset \mathcal{R}(G)$ consisting of irreducible representations is homeomorphic to $\mathcal{M}(G)_{ss}$, the smooth locus of $\mathcal{M}(G)$ consisting of stable and simple objects.

Of course we have statements corresponding to Propositions 5.8 and 5.9 for the moduli spaces of representations $\mathcal{R}(G^\theta)$, $\mathcal{R}(G^\sigma)$, $\mathcal{R}(G_\theta)$, and $\mathcal{R}(G_\sigma)$ (these can be proved directly or invoking Theorem 6.4). We denote their images in $\mathcal{R}(G)$, respectively, by $\widetilde{\mathcal{R}}(G^\theta)$, $\widetilde{\mathcal{R}}(G^\sigma)$, $\widetilde{\mathcal{R}}(G_\theta)$, and $\widetilde{\mathcal{R}}(G_\sigma)$.

8.2. Fixed points. From Theorems 6.4 and 7.6 we have the following.

Theorem 8.3. *Let $a \in \text{Out}_2(G)$. Consider the involutions*

$$\begin{aligned}\widehat{\iota}(a, \pm) : \mathcal{R}(G) &\rightarrow \mathcal{R}(G) \\ \rho &\mapsto a^\pm(\rho).\end{aligned}$$

Then

(1)

$$\bigcup_{[\theta] \in \mathcal{C}^{-1}(a)} \widetilde{\mathcal{R}}(G^\theta) \subset \mathcal{R}(G)^{\widehat{\iota}(a, +)},$$

(2)

$$\mathcal{R}(G)_i^{\hat{\imath}(a,+)} \subset \bigcup_{[\theta] \in \mathcal{C}\mathcal{L}^{-1}(a)} \widetilde{\mathcal{R}}(G^\theta),$$

except for $a = 1$.

(3)

$$\bigcup_{[\sigma] \in \widehat{\mathcal{C}\mathcal{L}}^{-1}(a)} \widetilde{\mathcal{R}}(G^\sigma) \subset \mathcal{R}(G)^{\hat{\imath}(a,-)},$$

(4)

$$\mathcal{R}(G)_i^{\hat{\imath}(a,-)} \subset \bigcup_{[\sigma] \in \widehat{\mathcal{C}\mathcal{L}}^{-1}(a)} \widetilde{\mathcal{R}}(G^\sigma).$$

Remark 8.4. The particular case of Theorem 8.3 for $\hat{\imath}(a, -)$ gives the representation statement corresponding to Theorem 7.3. This case involves only the equivalence classes of real forms that are inner equivalent to the compact conjugation, that is the real forms of Hodge type.

To describe the fixed points of the involutions of $\mathcal{R}(G)$ involving also an element $\lambda \in \mathcal{R}(Z) = \text{Hom}(\pi_1(X), Z)$, recall the extensions (2.8) and (2.9). These define maps

$$\hat{\gamma}_\theta : \mathcal{R}(G_\theta) \rightarrow \mathcal{R}(\Gamma_\theta), \quad \text{and} \quad \hat{\gamma}_\sigma : \mathcal{R}(G_\sigma) \rightarrow \mathcal{R}(\Gamma_\sigma),$$

which assign to every $\rho \in \mathcal{R}(G_\theta)$ (resp. $\rho \in \mathcal{R}(G_\sigma)$) an invariant $\hat{\gamma}_\theta(\rho) \in \mathcal{R}(\Gamma_\theta) = \text{Hom}(\pi_1(X), \Gamma_\theta)$ (resp. $\hat{\gamma}_\sigma(\rho) \in \mathcal{R}(\Gamma_\sigma) = \text{Hom}(\pi_1(X), \Gamma_\sigma)$). We will denote by $\mathcal{R}_\gamma(G_\theta)$ (resp. $\mathcal{R}_\gamma(G_\sigma)$) the subvariety of $\mathcal{R}(G_\theta)$ (resp. $\mathcal{R}(G_\sigma)$) with fixed invariant γ , and by $\widetilde{\mathcal{R}}_\gamma(G_\theta)$ (resp. $\widetilde{\mathcal{R}}_\gamma(G_\sigma)$), its corresponding image in $\mathcal{R}(G)$.

From Propositions 2.15, 2.17 and 7.8, we also have injective homomorphisms

$$\hat{c}_\theta : \mathcal{R}(\Gamma_\theta) \rightarrow \mathcal{R}(Z), \quad \text{and} \quad \hat{c}_\sigma : \mathcal{R}(\Gamma_\sigma) \rightarrow \mathcal{R}(Z).$$

Theorem 8.5. *Let $a \in \text{Out}_2(G)$ and $\lambda \in \mathcal{R}(Z)$ such that $a(\lambda) = \lambda^{-1}$.*

Consider the involutions

$$\begin{aligned} \hat{\imath}(a, \lambda, \pm) : \mathcal{R}(G) &\rightarrow \mathcal{R}(G) \\ \rho &\mapsto \lambda a^\pm(\rho). \end{aligned}$$

Then

(1)

$$\bigcup_{[\theta] \in \mathcal{C}\mathcal{L}^{-1}(a), \hat{c}_\theta(\gamma) = \lambda} \widetilde{\mathcal{R}}_\gamma(G_\theta) \subset \mathcal{R}(G)^{\hat{\imath}(a, \lambda, +)},$$

(2)

$$\mathcal{R}(G)_i^{\hat{\imath}(a, \lambda, +)} \subset \bigcup_{[\theta] \in \mathcal{C}\mathcal{L}^{-1}(a), \hat{c}_\theta(\gamma) = \lambda} \widetilde{\mathcal{R}}_\gamma(G_\theta),$$

except for $a = 1$ and $\lambda = 1$,

$$(3) \quad \bigcup_{[\sigma] \in \widehat{\mathcal{C}}^{-1}(a), \hat{c}_\sigma(\gamma)=\lambda} \widetilde{\mathcal{R}}_\gamma(G_\sigma) \subset \mathcal{R}(G)^{\widehat{\iota}(a,\lambda,-)},$$

$$(4) \quad \mathcal{R}(G)_i^{\widehat{\iota}(a,\lambda,-)} \subset \bigcup_{[\sigma] \in \widehat{\mathcal{C}}^{-1}(a), \hat{c}_\sigma(\gamma)=\lambda} \widetilde{\mathcal{R}}_\gamma(G_\sigma).$$

Remark 8.6. One can make corresponding considerations to those in Remarks 7.13, 7.14 and 7.15 for the involutions on $\mathcal{R}(G)$ studied in this section.

8.3. Hyperkähler and Lagrangian subvarieties of $\mathcal{M}(G)$. The fixed points on the smooth locus of $\mathcal{M}(G)$ of the involutions studied in Section 7 provide examples of hyperkähler and Lagrangian subvarieties of $\mathcal{M}(G)$. Recall from Section 4.3 that the smooth locus $\mathcal{M}(G)_{ss} \subset \mathcal{M}(G)$ has a hyperkähler structure, obtained as a hyperkähler quotient by solving Hitchin equations. In particular $\mathcal{M}(G)_{ss}$ has complex structures J_i , $i = 1, 2, 3$ satisfying the quaternion relations $J_i^2 = -I$, and real symplectic structures ω_i , $i = 1, 2, 3$. The Lagrangian condition we are referring above is with respect to the J_1 -holomorphic symplectic form $\Omega_1 = \omega_2 + \sqrt{-1}\omega_3$. More precisely we have the following.

Theorem 8.7. *Let $a \in \text{Out}_2(G)$ and $\alpha \in H^1(X, Z)$, such that $a(\alpha) = \alpha^{-1}$. Then, for every $\theta \in \text{Aut}_2(G)$ and $\gamma \in H^1(X, \Gamma_\theta)$ such that $[\theta] \in \mathcal{C}^{-1}(a)$, and $c_\theta(\gamma) = \alpha$, we have*

(1) $\mathcal{M}(G)_{ss} \cap \widetilde{\mathcal{M}}_\gamma(G_\theta, +)$ is a hyperkähler submanifold of $\mathcal{M}(G)_{ss}$. In particular $\mathcal{M}(G)_{ss} \cap \widetilde{\mathcal{M}}(G^\theta, +)$ is a hyperkähler submanifold of $\mathcal{M}(G)_{ss}$.

(2) $\mathcal{M}(G)_{ss} \cap \widetilde{\mathcal{M}}_\gamma(G_\theta, -)$ is a (J_1, Ω_1) -complex Lagrangian submanifold of $\mathcal{M}(G)_{ss}$. In particular $\mathcal{M}(G)_{ss} \cap \widetilde{\mathcal{M}}(G^\theta, -)$ is a (J_1, Ω_1) -complex Lagrangian submanifold of $\mathcal{M}(G)_{ss}$.

Proof. From Proposition 8.2, the involution $\iota(a, \alpha, +)$ on $\mathcal{M}(G)$ is holomorphic with respect to complex structure J_2 (the natural complex structure on $\mathcal{R}(G)$). Since it is J_1 -holomorphic (recall that J_1 is the natural complex structure on $\mathcal{M}(G)$) is also J_3 -holomorphic, and hence (1) follows.

The proof of (2) follows from the fact that the involution $\iota(a, \alpha, -)$ on $\mathcal{M}(G)$ is J_1 -holomorphic and J_2 -antiholomorphic, by Proposition 8.2, and hence J_3 -antiholomorphic. Since it is an isometry this implies that ω_2 , and ω_3 , and hence Ω_1 vanish on the fixed point locus, proving the assertion. This argument is given by Hitchin in [24] to give this result when $G = \text{SL}(2, \mathbb{C})$, and more generally in [14, 15]. \square

9. INVOLUTIONS OF $\mathcal{M}(\text{SL}(n, \mathbb{C}))$

To illustrate our main results, we will consider the case $G = \text{SL}(n, \mathbb{C})$. For this group, like for all classical groups, it is convenient to consider G -Higgs bundles in terms of vector bundles. From this point of view, a Higgs bundle over X is a pair (V, φ) , where V is a holomorphic vector bundle over X and $\varphi \in H^0(X, \text{End}(V) \otimes K)$, that is a homomorphism $\varphi : V \rightarrow V \otimes K$. In this case stability is defined in terms of slopes. Recall that the **slope** of a bundle is defined as $\mu(V) = \deg V / \text{rank } V$. We say that (V, φ) is stable if for every proper subbundle $V' \subset V$ such that $\varphi(V') \subset V'$ we have $\mu(V') < \mu(V)$. The Higgs bundle

is said to be polystable if $(V, \varphi) = \oplus_i (V_i, \varphi_i)$, with (V_i, φ_i) stable and $\mu(V_i) = \mu(V)$ for every i . This is the notion introduced in the original paper by Hitchin [24].

In order for a Higgs bundle (V, φ) to correspond to a $\mathrm{SL}(n, \mathbb{C})$ -Higgs bundle we must require that $\det V$ be trivial and $\mathrm{Tr}(\varphi) = 0$. The stability conditions for the principal and vector bundle points of view coincide naturally.

We will start with the simplest possible situation: $G = \mathrm{SL}(2, \mathbb{C})$.

9.1. $G = \mathrm{SL}(2, \mathbb{C})$. In this case $\mathrm{Out}(G) = \{1\}$, and the compact real form $\mathrm{SU}(2)$ and split real form $\mathrm{SL}(2, \mathbb{R}) \cong \mathrm{SU}(1, 1)$ of $\mathrm{SL}(2, \mathbb{C})$, corresponding to conjugations $\tau(A) = (\overline{A}^t)^{-1}$ and $\sigma_s(A) = \overline{A}$, respectively, are indeed inner equivalent. We can see this explicitly at the level of Lie algebras for example, since the conjugation with respect to the real form $\mathfrak{su}(2)$, $\tau(A) = -\overline{A}^t$, and the conjugation with respect to the real form $\mathfrak{sl}(2, \mathbb{R})$, $\sigma_s(A) = \overline{A}$, are related by

$$\sigma_s(A) = J\tau(A)J^{-1}$$

for $J \in \mathfrak{sl}(2, \mathbb{R})$ given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

simply because for every $A \in \mathfrak{sl}(2, \mathbb{R})$, one has $JA = -A^tJ$.

We consider now the involutions $\iota(a, \pm)$ defined in Section 7.2. Since $\iota(1, +)$ is the identity map, the only non-trivial involution of this type is $\iota(1, -)$ — the special case studied in Section 7.1. This case is studied by Hitchin in [24] (see also [19]).

The elements in $\mathrm{Aut}_2(G)$ corresponding to the conjugations τ and σ_c are given, respectively by $\theta_c = \tau^2 = \mathrm{Id}_G$ and $\theta_s = \sigma_c\tau$. We thus have that $\theta_s(A) = (A^t)^{-1}$, and hence $G^{\theta_c} = G = \mathrm{SL}(2, \mathbb{C})$ and $G^{\theta_s} = \mathrm{SO}(2, \mathbb{C}) \cong \mathbb{C}^*$.

The moduli space $\mathcal{M}(\mathrm{SL}(2, \mathbb{C}), -)$ is then isomorphic to the moduli space of polystable $\mathrm{SL}(2, \mathbb{C})$ -bundles, since $\mathfrak{g}^- = 0$ in this case, and hence the Higgs fields must vanish identically. By the theorem of Narasimhan and Seshadri [34] this is homeomorphic to $\mathcal{R}(\mathrm{SU}(2))$.

On the other hand the moduli space $\mathcal{M}(\mathrm{SO}(2, \mathbb{C}), -)$ is described by the isomorphism class of Higgs bundles of the form

$$(9.1) \quad V = L \oplus L^{-1} \quad \text{and} \quad \varphi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix},$$

where L is a line bundle, $\beta \in H^0(X, L^2 \otimes K)$ and $\gamma \in H^0(X, L^{-2} \otimes K)$. From stability one deduces (see [24]) that if $d = \deg L$, then $|d| \leq g - 1$, where g is the genus of X . The subspace of elements of $\mathcal{M}(\mathrm{SO}(2, \mathbb{C}), -)$ with fixed d satisfying this inequality defines a connected component, as shown in [24], where an explicit description of this subspaces is given.

The moduli space $\mathcal{M}(\mathrm{SO}(2, \mathbb{C}), -)$ is homeomorphic to $\mathcal{R}(\mathrm{SL}(2, \mathbb{R}))$. From the point of view of representations the inequality $|d| \leq g - 1$ is proved by Milnor [32] and the connectedness fixing the degree is proved by Goldman [21], who also shows that the components with maximal degree d is identified with Teichmüller space.

In the image of $\mathcal{M}(\mathrm{SO}(2, \mathbb{C}), -)$ in $\mathcal{M}(\mathrm{SL}(2, \mathbb{C}), -)$, denoted by $\widetilde{\mathcal{M}}(\mathrm{SO}(2, \mathbb{C}), -)$, the degree d component with $d \neq 0$ is identified with the degree $-d$ component, while in the degree 0 component there are some identifications.

Note that, although $\iota(1, +)$ is the identity, the moduli space $\mathcal{M}(\mathrm{SO}(2, \mathbb{C}))$, consisting of Higgs bundles of the form

$$(9.2) \quad V = L \oplus L^{-1} \quad \text{and} \quad \varphi = \begin{pmatrix} \psi & 0 \\ 0 & \psi \end{pmatrix},$$

where L is a line bundle and $\psi \in H^0(X, K)$, maps to $\mathcal{M}(\mathrm{SL}(2, \mathbb{C}))$, going to the strictly polystable locus.

We consider now the involutions studied in Section 7.3. In our situation $Z = Z(G) = \{\pm I\} \cong \mathbb{Z}/2$, and hence $H^1(X, Z) = J_2$, the 2-torsion elements in the Jacobian J of X . A direct approach to this is given in [19].

Consider the normalizer $NSO(2, \mathbb{C})$ of $SO(2, \mathbb{C})$ in $SL(2, \mathbb{C})$. This is generated by $SO(2, \mathbb{C})$ and $J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. The group generated by J is isomorphic to $\mathbb{Z}/4$ and fits in the exact sequence

$$(9.3) \quad 0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 1,$$

where the subgroup $\mathbb{Z}/2 \subset \mathbb{Z}/4$ is $\{\pm I\}$.

We thus have an exact sequence

$$(9.4) \quad 0 \longrightarrow SO(2, \mathbb{C}) \longrightarrow NSO(2, \mathbb{C}) \longrightarrow \mathbb{Z}/2 \longrightarrow 1.$$

Of course this is the sequence defining the Weyl group of $SL(2, \mathbb{C})$.

Similarly, we also have that $NSL(2, \mathbb{R})$, the normalizer of $SL(2, \mathbb{R})$ in $SL(2, \mathbb{C})$, is given by

$$(9.5) \quad 0 \longrightarrow SL(2, \mathbb{R}) \longrightarrow NSL(2, \mathbb{R}) \longrightarrow \mathbb{Z}/2 \longrightarrow 1.$$

Recall from Section 2.4 that $G_{\theta_s} = NSO(2, \mathbb{C})$, $G_{\sigma_s} = NSL(2, \mathbb{R})$, and hence $\Gamma_{\theta_s} = \Gamma_{\sigma_s} = \mathbb{Z}/2$. Since $Z = Z(G) = \mathbb{Z}/2$ we have that the maps c_{θ_s} and \hat{c}_{σ_s} intervening in Theorems 7.11 and 8.5 are isomorphisms. On the other hand $\Gamma_\tau = \{1\}$.

For every $\alpha \in H^1(X, Z) = J_2$ we consider the involutions $\iota(\alpha, \pm)$ of $\mathcal{M}(\mathrm{SL}(2, \mathbb{C}))$ defined by $(V, \varphi) \mapsto (V \otimes \alpha, \pm \varphi)$. From Theorems 7.11 and 8.5 we see that the fixed point locus of $\iota(\alpha, +)$ (with $\alpha \neq 1$) is described by the moduli space $\mathcal{M}_\alpha(NSO(2, \mathbb{C}), +)$, which in turn is homeomorphic to $\mathcal{R}_\alpha(NSO(2, \mathbb{C}))$; while the fixed points of $\iota(\alpha, -)$ are described by the moduli space $\mathcal{M}_\alpha(NSO(2, \mathbb{C}), -)$, homeomorphic to $\mathcal{R}_\alpha(NSL(2, \mathbb{R}))$. In the latter case α is allowed to be 1, and in that case the fixed point locus is described by $\mathcal{M}(SO(2, \mathbb{C}), -)$, homeomorphic to $\mathcal{R}(SL(2, \mathbb{R}))$.

One can construct the moduli spaces $\mathcal{M}_\alpha(\mathrm{NSO}(2, \mathbb{C}), \pm)$ in terms of Prym varieties: If α is a non-trivial element of J_2 , there is associated to it a canonical 2-sheeted étale cover $\pi : X_\alpha \rightarrow X$. Consider the norm homomorphism $\mathrm{Nm} : \mathrm{Pic}(X_\alpha) \rightarrow \mathrm{Pic}(X)$. Its kernel consists of two components and the one that contains the trivial bundle is the **Prym variety** P_α associated to α . If L is a line bundle on X_α , its direct image $\pi_* L$ is a rank two vector bundle V . Moreover V is polystable [33]. Since $\det E = \mathrm{Nm}(L) \otimes \alpha$, we must take line bundles in $S_\alpha = \mathrm{Nm}^{-1}(\alpha)$. This consists of two cosets of P_α , each of which is left invariant under the Galois involution. In [19] we give a description of the moduli spaces $\mathcal{M}_\alpha(\mathrm{NSO}(2, \mathbb{C}), \pm)$ in terms of S_α .

9.2. $G = \mathrm{SL}(n, \mathbb{C})$, $n > 2$. In this case $\mathrm{Out}(G) = \mathbb{Z}/2$. There are hence two cliques: $a = 1$ and $a = -1$. The classes in $\mathrm{Conj}(G)/\sim$ corresponding to the trivial clique $a = 1$ are represented (see [23] e.g.) by the conjugations $\sigma_{p,q}$, with $0 \leq p \leq q$ and $p + q = n$ given by

$$\begin{aligned} \sigma_{p,q} : \mathrm{SL}(n, \mathbb{C}) &\rightarrow \mathrm{SL}(n, \mathbb{C}) \\ A &\mapsto -I_{p,q}(\overline{A}^t)^{-1}I_{p,q}, \end{aligned}$$

where

$$(9.6) \quad I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

One has $G^{\sigma_{p,q}} = \mathrm{SU}(p, q)$. In particular $\tau := \sigma_{0,n}$ gives the compact real form $\mathrm{SU}(n)$. The elements in $\mathrm{Aut}_2(G)$ corresponding to $\sigma_{p,q}$ are given by $\theta_{p,q} := \tau\sigma_{p,q}$, and hence $G^{\theta_{p,q}} = \mathrm{S}(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}))$.

From Theorem 6.4 we have homeomorphisms

$$\mathcal{M}(G^{\theta_{p,q}}, +) \cong \mathcal{R}(\mathrm{S}(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}))) \quad \text{and} \quad \mathcal{M}(G^{\theta_{p,q}}, -) \cong \mathcal{R}(\mathrm{SU}(p, q)).$$

The moduli spaces $\mathcal{M}(G^{\theta_{p,q}}, -)$ and the correspondence with $\mathcal{R}(\mathrm{SU}(p, q))$ have been extensively studied in [7, 8], where there is counting of the number of connected components in terms of the so-called **Toledo invariant**, an integer invariant similar to the one appearing above for $\mathrm{SU}(1, 1)$. Of course $\mathcal{M}(G^{\theta_{0,n}}, -)$ is the moduli space of polystable $\mathrm{SL}(n, \mathbb{C})$ -bundles and the homeomorphism with $\mathcal{R}(\mathrm{SU}(n))$ is given by the Narasimhan–Seshadri theorem [34].

The involutions $\iota(-1, \pm)$ on $\mathcal{M}(\mathrm{SL}(n, \mathbb{C}))$ for the outer clique $a = -1$ is given by

$$\begin{aligned} \iota(-1, \pm) : \mathcal{M}(\mathrm{SL}(n, \mathbb{C})) &\rightarrow \mathcal{M}(\mathrm{SL}(n, \mathbb{C})) \\ (V, \varphi) &\mapsto (V^*, \mp \varphi^t), \end{aligned}$$

where V^* is the dual vector bundle and φ^t is the dual of φ tensored with the identity of K .

To describe the fixed points of $\iota(-, \pm)$ given by Theorem 7.6, we recall (see [23] e.g.) that the classes in $\mathrm{Conj}(G)/\sim$ corresponding to the outer clique $a = -1$ are represented by the conjugations $\sigma_s(A) = \overline{A}$, corresponding to the **split** real form, and if $n = 2m$, to $\sigma_*(A) = J_m \overline{A} J_m^{-1}$, where

$$J_m = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

We have the corresponding elements in $\text{Aut}_2(G)$ given by $\theta_s := \tau\sigma$ and $\theta_* := \tau\sigma_*$, and hence $G^{\sigma_s} = \text{SL}(n, \mathbb{R})$, $G^{\theta_s} = \text{SO}(n, \mathbb{C})$, and if $n = 2m$, $G^{\sigma_*} = \text{SU}^*(2m)$, and $G^{\theta_*} = \text{Sp}(2m, \mathbb{C})$.

As above, from Theorem 6.4 we have homeomorphisms

$$\mathcal{M}(G^{\theta_s}, +) \cong \mathcal{R}(\text{SO}(n, \mathbb{C})) \quad \text{and} \quad \mathcal{M}(G^{\theta_s}, -) \cong \mathcal{R}(\text{SL}(n, \mathbb{R})),$$

and

$$\mathcal{M}(G^{\theta_*}, +) \cong \mathcal{R}(\text{Sp}(2m, \mathbb{C})) \quad \text{and} \quad \mathcal{M}(G^{\theta_*}, -) \cong \mathcal{R}(\text{SU}^*(2m)),$$

The correspondence $\mathcal{M}(G^{\theta_s}, -) \cong \mathcal{R}(\text{SL}(n, \mathbb{R}))$ is studied by Hitchin in [26], where he counts the number of connected components of $\mathcal{R}(\text{SL}(n, \mathbb{R}))$ and introduces what he calls the higher Teichmüller components, now known as **Hitchin components**, components analogous to the Teichmüller components for $\text{SL}(2, \mathbb{R})$ mentioned above, and that exist for the split real form of every semisimple complex Lie group G . The correspondence $\mathcal{M}(G^{\theta_*}, -) \cong \mathcal{R}(\text{SU}^*(2m))$ is studied in [17], where it is shown that $\mathcal{R}(\text{SU}^*(2m))$ is connected.

The case of the involution $\iota(-, \pm)$ provides with a very good example to illustrate the need for restricting to the smooth locus $\mathcal{M}(G)_{ss}$ of $\mathcal{M}(G)$ in statement (2) of Theorem 7.6 (see Remark 7.14). Indeed, if an $\text{SL}(n, \mathbb{C})$ -Higgs bundle (V, φ) is not stable (which in this case implies also simple), the Higgs bundle is polystable and hence $(V, \varphi) = \oplus (V_i, \varphi_i)$ with (V_i, φ_i) stable and $\deg V_i = 0$, i.e. the structure group of (V, φ) reduces to $\text{S}(\Pi_i \text{GL}(n_i, \mathbb{C}))$ with $\sum n_i = n$. On each summand the involution $\iota(-, \pm)$ sends $(V_i, \varphi_i) \mapsto (V_i^*, \mp \varphi_i^t)$, implying in particular that, if (V, φ) is a fixed point of the involution, the bundles V_i reduce their structure group to $\text{O}(n_i, \mathbb{C})$ or $\text{Sp}(n_i, \mathbb{C})$. But since there is no need for all the bundles V_i simultaneously to be orthogonal or symplectic, the object (V, φ) may not be included in $\widetilde{\mathcal{M}}(G^\theta, \pm)$ for $\theta = \theta_s$ or $\theta = \theta_*$. In contrast with this, in the case of the involution $\iota(1, -)$, which sends $(V, \varphi) \mapsto (V, -\varphi)$, the inclusion (2) of Theorem 7.6 extends to the whole moduli space $\mathcal{M}(G)$ and not just the smooth locus.

Let $\alpha \in J_2(X)$. We consider now the involutions $\iota(1, \alpha, \pm)$ given by $(V, \varphi) \mapsto (V \otimes \alpha, \pm \varphi)$. To describe the fixed points, we have to compute the groups $G_{\theta_{p,q}}$, given by (2.9). A computation shows that $\Gamma_{\theta_{p,q}} = \{1\}$ if $p \neq q$ and $\Gamma_{\theta_{p,q}} \cong \mathbb{Z}/2$ if $p = q$ with $n = 2p$. We have a situation similar to that of $\text{SL}(2, \mathbb{R})$ in the previous section. We thus have exact sequences for the normalizers in $\text{SL}(n, \mathbb{C})$ of $\text{S}(\text{GL}(p, \mathbb{C}) \times \text{GL}(p, \mathbb{C}))$ and $\text{SU}(p, p)$, respectively given by

$$(9.7) \quad 0 \longrightarrow \text{S}(\text{GL}(p, \mathbb{C}) \times \text{GL}(p, \mathbb{C})) \longrightarrow \text{NS}(\text{GL}(p, \mathbb{C}) \times \text{GL}(p, \mathbb{C})) \longrightarrow \mathbb{Z}/2 \longrightarrow 1,$$

and

$$(9.8) \quad 0 \longrightarrow \text{SU}(p, p) \longrightarrow \text{NSU}(p, p) \longrightarrow \mathbb{Z}/2 \longrightarrow 1.$$

Since $Z = Z(G) = \mathbb{Z}/2p$ we have that the map $c_{\theta_{p,p}} : H^1(X, \Gamma_{\sigma_{p,p}}) \rightarrow H^1(X, Z)$ intervening in Theorem 7.11 is given by the injection $J_2(X) \hookrightarrow J_{2p}(X)$. Similarly for the map $\hat{c}_{\sigma_{p,p}}$ intervening in Theorem 8.5. From Theorems 7.11 and 8.5 we see that the fixed point locus of $\iota(1, \alpha, +)$ (with $\alpha \neq 1$) is described by the moduli space $\mathcal{M}_\alpha(\text{NS}(\text{GL}(p, \mathbb{C}) \times \text{GL}(p, \mathbb{C})), +)$, which in turn is homeomorphic to $\mathcal{R}_\alpha(\text{NS}(\text{GL}(p, \mathbb{C}) \times \text{GL}(p, \mathbb{C})))$. The fixed points of

$\iota(1, \alpha, -)$ are described by the moduli space $\mathcal{M}_\alpha(NS(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(p, \mathbb{C})), -)$ homeomorphic to $\mathcal{R}_\alpha(NSU(p, p))$. In the latter case α is allowed to be 1, and in that case the fixed point locus is described by $\mathcal{M}(S(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(p, \mathbb{C})), -)$ homeomorphic to $\mathcal{R}(SU(p, p))$.

As in the $n = 2$ case, one can describe the moduli spaces $\mathcal{M}_\alpha(NS(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(p, \mathbb{C})), \pm)$ in terms of certain objects in the 2-sheeted étale cover $\pi : X_\alpha \rightarrow X$ defined by α . This involves now **generalised Prym varieties** in the sense of Narasimhan–Ramanan (see [33, 20]).

There are no fixed points for the involutions $\iota(-1, \alpha, \pm)$ since $\Gamma_{\theta_s} = \Gamma_{\theta_*} = \{1\}$.

A direct approach to the study of the involutions $\iota(a, \pm)$ for $\mathrm{SL}(n, \mathbb{C})$ is carried out in [14].

10. HIGHER ORDER AUTOMORPHISMS OF $\mathcal{M}(G)$

We consider now arbitrary order n automorphisms of $\mathcal{M}(G)$ defined by elements in $H^1(X, Z) \rtimes \mathrm{Out}(G) \times \mathbb{C}^*$. Let $\theta \in \mathrm{Aut}_n(G)$ and $\zeta_k := \exp(2\pi i \frac{k}{n})$, with $0 \leq k \leq n-1$. Consider the moduli spaces $\mathcal{M}(G^\theta, \zeta_k)$ and $\mathcal{M}(G_\theta, \zeta_k)$ defined in Section 5.3. As in the $n = 2$ case, we denote by $\widetilde{\mathcal{M}}(G^\theta, \zeta_k)$ and $\widetilde{\mathcal{M}}(G_\theta, \zeta_k)$ the images of $\mathcal{M}(G^\theta, \zeta_k)$ and $\mathcal{M}(G_\theta, \zeta_k)$ in $\mathcal{M}(G)$, respectively. The following is a straightforward generalisation of Theorem 7.6.

Theorem 10.1. *Let $a \in \mathrm{Out}_n(G)$. Consider the automorphism*

$$\begin{aligned} \iota(a, \zeta_k) : \mathcal{M}(G) &\rightarrow \mathcal{M}(G) \\ (E, \varphi) &\mapsto (a(E), \zeta_k a(\varphi)). \end{aligned}$$

Then

(1)

$$\bigcup_{[\theta] \in \mathcal{C}_n^{-1}(a)} \widetilde{\mathcal{M}}(G^\theta, \zeta_k) \subset \mathcal{M}(G)^{\iota(a, \zeta_k)},$$

(2)

$$\mathcal{M}(G)_{ss}^{\iota(a, \zeta_k)} \subset \bigcup_{[\theta] \in \mathcal{C}_n^{-1}(a)} \widetilde{\mathcal{M}}(G^\theta, \zeta_k)$$

(except for $\iota(1, 1)$),

where $\mathcal{C}_n : \mathrm{Aut}_n(G)/\sim \rightarrow \mathrm{Out}_n(G)$ is defined in Proposition 2.22, and $\mathcal{C}_n^{-1}(a)$ is, by Proposition 2.24, in bijection with $H_a^1(\mathbb{Z}/n, \mathrm{Ad}(G))$.

Remark 10.2. The case $a = 1$ in Theorem 10.1 reduces to the study of **cyclic** Higgs bundles done by Simpson in [48] for $\mathrm{SL}(n, \mathbb{C})$, and recently developed by Collier in general [10]. It should be interesting to compare the different points of view.

Consider the map

$$\gamma_\theta : H^1(X, \underline{G}_\theta) \longrightarrow H^1(X, \Gamma_\theta)$$

induced by the homomorphism $G_\theta \longrightarrow \Gamma_\theta$. To every element $(E, \varphi) \in \mathcal{M}(G_\theta, \zeta_k)$ in $\mathcal{M}(G)$ we can associate an invariant $\gamma = \gamma_\theta(E)$, and consider the subvariety $\mathcal{M}_\gamma(G_\theta, \zeta_k)$ of objects

with fixed invariant γ , and its image $\widetilde{\mathcal{M}}_\gamma(G_\theta, \zeta_k)$ in $\mathcal{M}(G)$. Let

$$c_\theta : H^1(X, \Gamma_\theta) \rightarrow H^1(X, Z)$$

be the map induced by the injective homomorphism $\tilde{c} : \Gamma_\theta \rightarrow Z$ defined in Proposition 2.26. The following is a straightfroward extension of Theorem 7.11.

Theorem 10.3. *Let $(\alpha, a) \in (H^1(X, Z) \rtimes \text{Out}(G))_n$. Consider the automorphism*

$$\begin{aligned} \iota(a, \alpha, \zeta_k) : \mathcal{M}(G) &\rightarrow \mathcal{M}(G) \\ (E, \varphi) &\mapsto (a(E) \otimes \alpha, \zeta_k a(\varphi)). \end{aligned}$$

Then

(1)

$$\bigcup_{[\theta] \in \mathcal{C}_n^{-1}(a), c_\theta(\gamma) = \alpha} \widetilde{\mathcal{M}}_\gamma(G_\theta, \zeta_k) \subset \mathcal{M}(G)^{\iota(a, \alpha, \zeta_k)}.$$

(2)

$$\mathcal{M}(G)_{ss}^{\iota(a, \alpha, \zeta_k)} \subset \bigcup_{[\theta] \in \mathcal{C}_n^{-1}(a), c_\theta(\gamma) = \alpha} \widetilde{\mathcal{M}}_\gamma(G_\theta, \zeta_k)$$

(except for $\iota(1, 1, 1)$),

where $\mathcal{C}_n : \text{Aut}_n(G)/\sim \rightarrow \text{Out}_n(G)$ is defined in Proposition 2.22, and $\mathcal{C}_n^{-1}(a)$ is, by Proposition 2.24, in bijection with $H_a^1(\mathbb{Z}/n, \text{Ad}(G))$.

11. $\text{Spin}(8, \mathbb{C})$ -HIGGS BUNDLES AND TRIALITY

In this section we give an application of Theorems 7.3, 7.6 8.3 and 10.1 to the case $G = \text{Spin}(8, \mathbb{C})$. This is the (simply connected) simple complex Lie group with the largest group of outer automorphisms, namely $\text{Out}(G) = S_3$ (see Table 1), thus exhibiting very interesting phenomena.

11.1. Involutions. $G = \text{Spin}(8, \mathbb{C})$ fits in the exact sequence

$$(11.1) \quad 1 \longrightarrow \mathbb{Z}/2 \longrightarrow \text{Spin}(8, \mathbb{C}) \longrightarrow \text{SO}(8, \mathbb{C}) \longrightarrow 1,$$

and hence its Lie algebra is $\mathfrak{so}(8, \mathbb{C})$. The set (2.3) of isomorphism classes of conjugations for $\mathfrak{g} = \mathfrak{so}(8, \mathbb{C})$ is represented (see e.g. [23]) by the conjugations

$$\sigma^{p,q}(A) = I_{p,q} \overline{A} I_{p,q} \text{ with } 0 \leq p \leq q \text{ and } p + q = 8,$$

where $I_{p,q}$ is given by (9.6), corresponding to the real forms $\mathfrak{so}(p, q)$, and

$$\sigma^*(A) = J_4 \overline{A} J_4^{-1},$$

where

$$J_4 = \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix},$$

corresponding to the real form $\mathfrak{so}^*(8)$. The compact conjugation is given by $\tau = \sigma^{0,8}$ and the elements in $\text{Aut}_2(G)$ corresponding to the conjugations above are $\theta^{p,q} = \tau\sigma^{p,q}$ and $\theta^* = \tau\sigma^*$ and thus given by

$$(11.2) \quad \theta^{p,q}(A) = I_{p,q} A I_{p,q} \text{ with } 0 \leq p \leq q \text{ and } p + q = 8,$$

and

$$(11.3) \quad \theta^*(A) = J_4 A J_4^{-1}.$$

Of course, all these conjugations and involutions can be lifted to $\text{Spin}(8, \mathbb{C})$ (denoted in the same way), leading to real forms $\text{Spin}_0(p, q)$ with $p + q = 8$ (see Remark 2.5) and $\text{Spin}^*(8)$. One can show that $\text{Spin}(8)$, $\text{Spin}_0(2, 6)$, $\text{Spin}_0(4, 4)$, and $\text{Spin}^*(8)$ are in the trivial clique, that is, are the real forms of Hodge type, while $\text{Spin}_0(1, 7)$ and $\text{Spin}_0(3, 5)$ are in a non-trivial clique, say $a_1 \in \text{Out}_2(G)$. If $b \in \text{Out}(G) = S^3$ is an element of order 3, the group S^3 can be generated with a_1 and b . The elements $a_2 := ba_1b^{-1}$ and $a_3 := b^2a_1b^{-2}$ have also order 2, and hence if B is a lift of b to $\text{Aut}_3(G)$, the real groups $B(\text{Spin}_0(1, 7))$ and $B(\text{Spin}_0(3, 5))$ and $B^2(\text{Spin}_0(1, 7))$ and $B^2(\text{Spin}_0(3, 5))$ are isomorphic to $\text{Spin}_0(1, 7)$ and $\text{Spin}_0(3, 5)$, respectively, but of course by outer isomorphisms. Among the complex simple Lie algebras, this is the only case for which there is this type of phenomenon. To distinguish these different subgroups of $\text{Spin}(8, \mathbb{C})$ we introduce the notation

$$\text{Spin}_0(1, 7)_i := B^{i-1}(\text{Spin}_0(1, 7)) \text{ with } i = 1, 2, 3,$$

and

$$\text{Spin}_0(3, 5)_i := B^{i-1}(\text{Spin}_0(3, 5)) \text{ with } i = 1, 2, 3.$$

These correspond to conjugations and holomorphic involutions $\sigma_i^{1,7} := B^{i-1}\sigma^{1,7}B^{1-i}$, $\theta_i^{1,7} := B^{i-1}\theta^{1,7}B^{1-i}$, and $\sigma_i^{3,5} := B^{i-1}\sigma^{3,5}B^{1-i}$, $\theta_i^{3,5} := B^{i-1}\theta^{3,5}B^{1-i}$, respectively, with $i = 1, 2, 3$.

The subgroup $G^{\theta^{p,q}}$ of fixed points of the involution $\theta^{p,q}$ given by (11.2) is $\text{Spin}(p, \mathbb{C}) \times \text{Spin}(q, \mathbb{C})$. The decomposition $\mathfrak{g} = \mathfrak{g}_{p,q}^+ \oplus \mathfrak{g}_{p,q}^-$ in (± 1) -eigenspaces for $\theta^{p,q}$ is given by

$$\mathfrak{g}_{p,q}^+ = \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \mid X \in \mathfrak{so}(p, \mathbb{C}), Y \in \mathfrak{so}(q, \mathbb{C}) \right\},$$

and

$$\mathfrak{g}_{p,q}^- = \left\{ \begin{pmatrix} 0 & Z \\ -Z^t & 0 \end{pmatrix} \mid Z \text{ complex } (p \times q)\text{-matrix} \right\}.$$

Clearly, for $p = 1, q = 7$ one has $G^{\theta_i^{1,7}} = B^{i-1}(\text{Spin}(1, \mathbb{C}) \times \text{Spin}(7, \mathbb{C}))$ and the (± 1) -eigenspace decomposition of \mathfrak{g} for $\theta_i^{1,7}$ is given

$$\mathfrak{g} = B^{i-1}(\mathfrak{g}_{1,7}^+) \oplus B^{i-1}(\mathfrak{g}_{1,7}^-),$$

where, as usual, we are denoting by the same letter the element in $\text{Aut}(\mathfrak{g})$ induced by an element in $\text{Aut}(G)$. Similarly for the other outer case corresponding to $p = 3, q = 5$.

The subgroup G^{θ^*} of fixed points of the involution (11.3) is the double cover determined by the exact sequence (11.1) of the subgroup $\text{GL}(4, \mathbb{C}) \subset \text{SO}(8, \mathbb{C})$ given by elements

$$\begin{pmatrix} A & 0 \\ & (A^t)^{-1} \end{pmatrix},$$

where $A \in \mathrm{GL}(4, \mathbb{C})$. The decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ in (± 1) -eigenspaces for θ^* is given by $\mathfrak{g}^+ = \mathfrak{gl}(4, \mathbb{C})$ and $\mathfrak{g}^- = \Lambda^2(\mathbb{C}^4) \oplus \Lambda^2(\mathbb{C}^4)^*$.

We have all the ingredients now to apply Theorems 7.3 and 7.6 to our situation.

Theorem 11.1. *Let $G = \mathrm{Spin}(8, \mathbb{C})$. Consider the involution*

$$\begin{aligned} \iota : \mathcal{M}(G) &\rightarrow \mathcal{M}(G) \\ (E, \varphi) &\mapsto (E, -\varphi). \end{aligned}$$

Then

$$(1) \quad \bigcup_{p=0,2,4; p+q=8} \widetilde{\mathcal{M}}(G^{\theta^{p,q}}, -) \bigcup \widetilde{\mathcal{M}}(G^{\theta^*}, -) \subset \mathcal{M}(G)^\iota,$$

$$(2) \quad \mathcal{M}(G)_{ss}^\iota \subset \bigcup_{p=0,2,4; p+q=8} \widetilde{\mathcal{M}}(G^{\theta^{p,q}}, -) \bigcup \widetilde{\mathcal{M}}(G^{\theta^*}, -)$$

Theorem 11.2. *Let $G = \mathrm{Spin}(8, \mathbb{C})$ and let $1 \neq a_i \in \mathrm{Out}_2(G)$ with $i = 1, 2, 3$. Consider the involutions*

$$\begin{aligned} \iota(a_i, \pm) : \mathcal{M}(G) &\rightarrow \mathcal{M}(G) \\ (E, \varphi) &\mapsto (a_i(E), \pm a_i(\varphi)). \end{aligned}$$

Then

$$(1) \quad \bigcup_{p=1,3; p+q=8} \widetilde{\mathcal{M}}(G_i^{\theta^{p,q}}, \pm) \subset \mathcal{M}(G)^{\iota(a_i, \pm)},$$

$$(2) \quad \mathcal{M}(G)_{ss}^{\iota(a_i, \pm)} \subset \bigcup_{p=1,3; p+q=8} \widetilde{\mathcal{M}}(G_i^{\theta^{p,q}}, \pm)$$

Of course we have the corresponding versions of Theorem 8.3 regarding the corresponding involutions in the moduli space of representations $\mathcal{R}(G)$.

It is clear that the subvarieties $\widetilde{\mathcal{M}}(G_i^{\theta^{p,q}}, \pm)$ for $p = 1, 3$ are transformed one into another by the action of the order 3 element $b \in \mathrm{Out}(G)$, namely,

$$\widetilde{\mathcal{M}}(G_i^{\theta^{p,q}}, \pm) = b^{i-1}(\widetilde{\mathcal{M}}(G^{\theta^{p,q}}, \pm)),$$

for $i = 1, 2, 3$.

11.2. Order 3 automorphisms. Let $b \in \mathrm{Out}_3(G)$, and let $\zeta_k := \exp(2\pi i \frac{k}{3})$, with $k = 0, 1, 2$. We study now the order 3 automorphism of $\mathcal{M}(G)$ given by $(E, \varphi) \mapsto (b(E), \zeta_k b(\varphi))$. To do this, we need to understand the 3-clique b , that is we need to know the set $\mathcal{C}_3^{-1}(b) \in \mathrm{Aut}_3(G)/\sim$. This is studied in [49]. It turns out that $\mathcal{C}_3^{-1}(b)$ consists of two classes. One can choose representatives $\theta_1, \theta_2 \in \mathrm{Aut}_3(G)$ of these two classes such that

$$G^{\theta_1} = \mathrm{PSL}(3, \mathbb{C}) \subset \mathrm{Spin}(8, \mathbb{C})$$

and the decomposition of \mathfrak{g} in ζ_k -eigenspaces for $k = 0, 1, 2$ is

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2 = \mathfrak{sl}(3, \mathbb{C}) \oplus S^2(\mathbb{C}^3) \oplus S^2(\mathbb{C}^3)^*,$$

where $S^2(\mathbb{C}^3)$ is the 2-symmetric tensor product of the fundamental representation of $\mathrm{SL}(3, \mathbb{C})$, which is thus 10-dimensional; and

$$G^{\theta_2} = G_2 \subset \mathrm{Spin}(8, \mathbb{C}),$$

and

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2 = \mathfrak{g}_2 \oplus \mathbb{C}^7 \oplus \mathbb{C}_7,$$

where \mathbb{C}^7 is the fundamental representation of G_2 .

We can now apply Theorem 10.1 to obtain the following.

Theorem 11.3. *Let $G = \mathrm{Spin}(8, \mathbb{C})$, b be a non-trivial element of order 3 in $\mathrm{Out}(G) = S^3$, and $\zeta_k := \exp(2\pi i \frac{k}{3})$, with $k = 0, 1, 2$. Consider the order 3 automorphism*

$$\begin{aligned} \iota(b, \zeta_k) : \mathcal{M}(G) &\rightarrow \mathcal{M}(G) \\ (E, \varphi) &\mapsto (b(E), \zeta_k b(\varphi)). \end{aligned}$$

Then

(1)

$$\widetilde{\mathcal{M}}(\mathrm{PSL}(3, \mathbb{C}), \zeta_k) \cup \widetilde{\mathcal{M}}(G_2, \zeta_k) \subset \mathcal{M}(G)^{\iota(b, \zeta_k)},$$

(2)

$$\mathcal{M}(G)_{ss}^{\iota(b, \zeta_k)} \subset \widetilde{\mathcal{M}}(\mathrm{PSL}(3, \mathbb{C}), \zeta_k) \cup \widetilde{\mathcal{M}}(G_2, \zeta_k).$$

A direct approach to this case has been given in [2], where the stability conditions for the $(\mathrm{PSL}(3, \mathbb{C}), \zeta_k)$ - and (G_2, ζ_k) -Higgs bundles have been explicitly worked out.

One has $b^{-1} = a_i b a_i$ for any $a \neq 1$ in $\mathrm{Out}_2(G)$, and hence if $A \in \mathrm{Aut}_2(G)$ is a lift of a , the 3-clique b^{-1} has $\theta'_1 = A\theta_1 A^{-1}$ and $\theta'_2 = A\theta_2 A^{-1}$ as representatives of the two classes in $\mathcal{C}_3^{-1}(b^{-1})$, thus having $G^{\theta'_1} = A(\mathrm{PSL}(3, \mathbb{C}))$ and $G^{\theta'_2} = A(G_2)$. The fixed points of $\iota(b, \zeta_k)$ are hence moved to the fixed points of $\iota(b^{-1}, \zeta_k)$ by the action of b on $\mathcal{M}(G)$.

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